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Branching through G_2

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Abstract. The purpose of this expository article is to make an introduction to representation theory for Lie groups/algebras by focusing on the branching of a representation from $Spin_7$ to SL_3 through subgroups $Spin_6$, and G_2.

1. Introduction

This manuscript is an expanded version of the lecture notes of the first author delivered during the 20^{th} Gökova Geometry-Topology Conference. The last part of the manuscript includes some calculations from the joint investigation on branching problems of both authors. However, the first author is responsible for all mistakes, typos, or any false claims that may appear in the manuscript.

Let Φ denote the configuration of vectors in $\mathbb{R}^2$ depicted as in Figure 1. To each of these twelve vectors in Φ, we assign variables: $\alpha_i \leftrightsquigarrow X_i$ for $i = 1, \dots, 6$, and $-\alpha_i \leftrightsquigarrow Y_i$ for $i = 1, \dots, 6$. In addition, we introduce two more variables H_1 and H_2 and form the 14-dimensional $\mathbb{C}$-vector space $\mathfrak{g}_2$ with basis

$$\mathcal{V} = \{X_1, X_2, X_3, X_4, X_5, X_6, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, H_1, H_2\}.$$

Our focus here is on a particular algebra structure on the space $\mathfrak{g}_2$ whose multiplicative structure is determined by Table 1 below, and the requirement that $A * B = -B * A$ for all $A, B \in \mathfrak{g}_2$. This multiplication table, essentially, is imposed by our careful choice of initial conditions a) $X_1 * Y_1 = H_1$, b) $X_2 * Y_2 = H_2$, and by the diagrammatic rule

$$Z_i * Z_j = c_{i,j} Z_k, \ c_{i,j} \in \mathbb{Z}$$

whenever $(Z_i, Z_j, Z_k) \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}$ corresponds to the triple $(\alpha_i, \alpha_j, \alpha_k) \in \Phi \times \Phi \times \Phi$ satisfying $\alpha_i + \alpha_j = \alpha_k$.

The overarching goal of our manuscript is to introduce the basic ideas of the representation theory of $\mathfrak{g}_2$ by studying the behavior of its modules under restrictions. To elaborate on it in more familiar terms, let us pass to the group setting: A *linear algebraic group* G is a closed subgroup of the general linear group of $n \times n$ invertible matrices over the field of complex numbers $\mathbb{C}$. The *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ of G is, by definition, the

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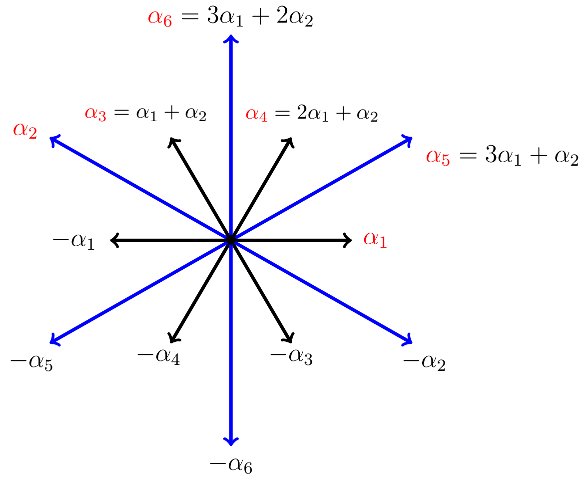


FIGURE 1. A configuration of twelve vectors from $\mathbb{R}^2$

$*$	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	Y_6
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	X_6	$-X_5$	0
X_3							$H_1 + 3H_2$	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1 + 3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											$H_1 + H_2$	0	$-Y_2$
Y_5												X_2	0
X_6													$H_1 + 2H_2$

TABLE 1. Multiplicative structure of $\mathfrak{g}_2$

linear space consisting of left-translation-invariant vector fields on G with multiplication given by $[X, Y] := XY - YX$. Although it is not immediate from its above definition, $\mathfrak{g}_2$ is the Lie algebra of a linear algebraic group, which we denote by G_2. Here, we investigate the interactions between G_2 and the some other well known classical groups, namely, SL_n (special linear group), SO_n (special orthogonal group), and $Spin_n$ (spin group; the universal covering of SO_n). More precisely, let V be a representation of $Spin_7$, meaning that V is a vector space and there exists a group homomorphism $\sigma : Spin_7 \to GL(V)$. We analyze V when it is viewed as an H-module for H one of the subgroups of $G = Spin_7$ that are listed in Figure 2. In general, understanding this kind of branching problem for an arbitrary pairs of groups (G, H) is a challenging problem, however, for certain pairs of classical groups satisfactory solutions exist. In our article, we make a modest introduction to this important topic in combinatorial representation theory, and while doing so, we hope that our article serves useful as a quick introduction to the Lie theory through small rank examples. To this end, we decided to split our paper into two parts, first of which summarizes the basic principles of Lie theory and related representation theory. In the second part of our paper, we present some well known branching theorems and their applications to the groups of Figure 2.

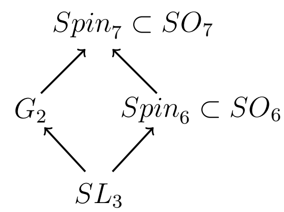


FIGURE 2. Two paths

We should mention that all of the branching computations we present in the last part of our paper have been performed in some form in the literature before. See, [GaSh] and [Wyb] as well as the references in these papers. However, we still hope that our resulting computations look somewhat cleaner and hopefully helpful for some other purposes.

2. Part I

2.1. Notation and Preliminaries

Throughout our paper, G denotes a linear algebraic group and we consider only finite dimensional representations of G. Furthermore, we restrict our attention to rational representations $\rho : G \to GL(V)$, that is to say, after choosing an ordered basis for the space V, the entries of all matrices $\rho(g) \in GL_m \cong GL(V)$ are rational functions of the co-ordinates on G (these are restrictions of the co-ordinate functions from GL_n).

A *regular function* on an algebraic variety X (hence on a linear algebraic group $X = G$) is a collection of pairs $(U, f/g)$, one for each point $x \in X$, where $x \in U \subset X$ is an open set and $f/g : U \to \mathbb{C}$ is a well defined rational function. Of course there has to be a compatibility condition on these pairs and it is a good exercise to figure it out. With respect to point-wise multiplication of the rational functions, the collection of all regular functions on X forms a ring. A *derivation* on the ring $\mathcal{O}[G]$ of regular functions on G is a linear map $\nu : \mathcal{O}[G] \to \mathcal{O}[G]$ such that $\nu(f_1 f_2) = \nu(f_1) f_2 + f_1 \nu(f_2)$ for $f_1, f_2 \in \mathcal{O}[G]$. The set of all derivations on $\mathcal{O}[G]$, which we denote by $\mathrm{Der}(G)$ forms a vector space. Then the *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ of G is defined to be the subspace $\mathfrak{g} \subset \mathrm{Der}(G)$ consisting of $\nu \in \mathrm{Der}(G)$ such that

$$\mathcal{L}_x \nu = \nu \mathcal{L}_x, \quad \text{for all} \quad x \in G$$

where $\mathcal{L}_x : \mathcal{O}[G] \to \mathcal{O}[G]$ is the "left-translation by x" defined by $\mathcal{L}_x(f)(y) = f(x^{-1}y)$ for $f \in \mathcal{O}[G]$ and $y \in G$. The multiplicative structure on $\mathfrak{g}$ is given by the "bracket" $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $[\nu, \mu] = \nu \circ \mu - \mu \circ \nu$, $\nu, \mu \in \mathfrak{g}$. Here, circle means composition of functions. When GL_n is viewed as the general linear group $GL(\mathbb{R}^{2n})$ over the field of real numbers, the group $G \subseteq GL_n$ becomes a Lie group and moreover its Lie algebra can be identified with the Lie algebra of left invariant vector fields on G in the sense of differentiable manifolds.

Abstractly, a *Lie algebra* is nothing more than a vector space $\mathfrak{g}$ endowed with a bilinear multiplication $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[x, x] = 0$ for all $x \in \mathfrak{g}$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Example 2.1. On $\mathfrak{g} = \mathbb{R}^3$ we have the "cross-product" of calculus: $e_1 \times e_2 = e_3$, $e_3 \times e_1 = e_2$, and $e_2 \times e_3 = e_1$, where e_1, e_2, e_3 are the standard basis elements. Remember that $e_i \times e_j = -e_j \times e_i$ for all $i, j = 1, 2, 3$. Extended by linearity to all of $\mathbb{R}^3$, it is easy to check that the cross product satisfies the defining properties of a Lie bracket.

Of course, there are many different Lie algebra structures on a fixed vector space for Lie brackets reflect the behavior of the underlying groups.

Example 2.2. The *Heisenberg algebra* is the three dimensional Lie algebra

$$H_3(\mathbb{R}) = \mathrm{Span}_{\mathbb{R}}\{x, y, z\} \cong \mathbb{R}^3$$

with respect to the Lie bracket $[x, y] = z,\ [x, z] = 0,\ [y, z] = 0$. Clearly, $H_3(\mathbb{R})$ is not isomorphic to the "cross-product algebra" from Example 2.1.

The Lie algebra of $GL(V)$ is denoted by $\mathfrak{gl}(V)$, which, as a vector space is isomorphic to the space of all $n \times n$ matrices. The bracket on $\mathfrak{gl}(V)$ is given by $[X, Y] = X \circ Y - Y \circ X$ for $X, Y \in \mathfrak{gl}(V)$.

A linear map $L : \mathfrak{g}_1 \to \mathfrak{g}_2$ between two Lie algebras is called a *Lie algebra homomorphism*, if $L([x, y]_{\mathfrak{g}_1}) = [L(x), L(y)]_{\mathfrak{g}_2}$ for all $x, y \in \mathfrak{g}$. It is not difficult to see that isomorphic linear algebraic groups have isomorphic Lie algebras. A *(linear) representation* of $\mathfrak{g}$ is a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathfrak{gl}(V)$ for some vector space V. The *adjoint representation* of $\mathfrak{g}$ is the representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto \mathrm{ad}\ x$, where $\mathrm{ad}\ x$ is the linear map defined by $\mathrm{ad}\ x(y) := [x, y]$, $y \in \mathfrak{g}$. The kernel of ad is called the *center* of $\mathfrak{g}$. If the center is all of $\mathfrak{g}$, then the Lie algebra is called *abelian.* The kernel of a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is called an *ideal* of $\mathfrak{g}$. Equivalently, a subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is called an *ideal*, if for every $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$, the element $[x, a] \in \mathfrak{g}$ lies in $\mathfrak{a}$. Observe that an ideal is a Lie subalgebra in a natural way. A non-abelian Lie algebra is called *simple*, if it does not posses an ideal other than itself and the zero subspace. $\mathfrak{g}$ is called *semi-simple*, if it decomposes into a direct sum of its simple ideals. We call an algebraic group G *semi-simple*, if its Lie algebra is semi-simple, G is called *(algebraically) simply-connected*, if any surjective group homomorphism $\sigma : H \to G$ with a finite kernel and H connected is an isomorphism of algebraic groups. Now, suppose G

is a simply-connected group, and its Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ is semi-simple. In this case, any connected algebraic group with Lie algebra $\mathfrak{g}$ is a quotient of G by a subgroup of the center of G. In particular, G is the largest algebraic group with Lie algebra $\mathfrak{g}$.

A module V for a Lie algebra $\mathfrak{g}$ is called irreducible if V contains precisely two submodules: $\{0\}$ and V itself. Two very important facts about irreducible representations are :

- *Schur's Lemma*: Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be an irreducible representation. Then the endomorphisms which commute with $\rho(\mathfrak{g})$ are the scaling linear transformations.
- *Weyl's Theorem*: Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be finite dimensional representation of a semi-simple Lie algebra. Then ρ is completely reducible. In other words, V is a direct sum of its irreducible $\mathfrak{g}$-submodules.

Finally, observe that the differential $\mathrm{d}\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ of a representation of an algebraic group $\pi : G \to GL(V)$ is a Lie algebra representation.

2.2. More on Lie algebras

The classification of semi-simple Lie algebras (over $\mathbb{C}$) was achieved by the joint efforts of Wilhelm Killing and Élie Cartan in the late 1800's. Before we summarize the details of the classification scheme, we make a brief interlude on "real forms." Given a complex semisimple Lie algebra (meaning that it is defined over $\mathbb{C}$) $\mathfrak{g}$, a real Lie algebra $\mathfrak{g}_0$ (meaning that it is defined over $\mathbb{R}$) is called a *real form* of $\mathfrak{g}$, if the complexified vector space $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathfrak{g}$. In general there are many real forms of a given complex Lie algebra. However, if $\mathfrak{g}$ is semisimple, there is a unique particularly important special case called the "compact real form" of $\mathfrak{g}$, which is characterized by a bilinear form as follows.

The *Killing form* of a Lie algebra $\mathfrak{h}$ defined over a field $\mathbb{K}$ is the unique symmetric bilinear form $B : \mathfrak{h} \times \mathfrak{h} \to \mathbb{K}$ defined by

$$B(x, y) := \mathrm{Tr}(\mathrm{ad}\ x \circ \mathrm{ad}\ y) = \mathrm{Tr}(\mathrm{ad}\ y \circ \mathrm{ad}\ x), \qquad x, y \in \mathfrak{h}.$$

A real form $\mathfrak{g}_0$ is called a *compact real form* for the complex semisimple Lie algebra $\mathfrak{g}$, if the associated Killing form of $\mathfrak{g}_0$ is negative-definite. The significance of a compact real form is that it is the Lie algebra of a compact (in the topological sense) real Lie group. Moreover, if G is a connected linear algebraic semisimple group over $\mathbb{C}$, then there is a one-to-one correspondence between the irreducible representations of G and the irreducible representations of the compact real form $\mathfrak{g}_0$ of the complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. This correspondence ties together the analysis on compact (real) Lie groups and the representation theory of connected linear algebraic groups over $\mathbb{C}$.

As it is noted above the Killing form is a bridge between complex and real Lie algebras. In our exposition we focus on complex Lie algebras only, however, the Killing form is still our primary tool for proving many structural results.

Lemma 2.1 (Cartan's criterion)**.** $\mathfrak{g}$ *is semi-simple if and only if its Killing form B is non-degenerate, that is to say, $B(x, y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$.*

Now we have a strategy for the classification of semi-simple Lie algebras:

(1) Choose a maximal subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acting semisimply on $\mathfrak{g}$, which is unique up to conjugacy by G.
(2) Using Killing's form attach to $\mathfrak{h}$ a finite system of vectors (called the "root system" of $(\mathfrak{g}, \mathfrak{h})$) in a Euclidean subspace of the dual vector space $\mathfrak{h}^*$.
(3) Classify all possible "irreducible" root systems.
(4) Show that there is a unique simple Lie algebra for each such root system.

It turns out that there are four countably-infinite families of irreducible root systems, labeled by A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, and D_n $(n \geq 4)$. We list below the corresponding Lie algebras, which are customarily called as the *classical Lie algebras*.

Type A_n (the special linear Lie algebra $\mathfrak{sl}_{n+1}$): Let $\mathfrak{gl}_{n+1}$ denote the Lie algebra of all complex $(n+1) \times (n+1)$ matrices with the Lie bracket $[x, y] = xy - yx$. Since

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) \quad \text{and} \quad \mathrm{Tr}(x+y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$$

for all $x, y \in \mathfrak{gl}_n$, we see that the vector subspace $\mathfrak{sl}_{n+1} \subset \mathfrak{gl}_{n+1}$ consisting of $x \in \mathfrak{gl}_{n+1}$ such that $\mathrm{Tr}(x) = 0$ is a Lie subalgebra. The unique simply-connected algebraic group with Lie algebra $\mathfrak{sl}_{n+1}$ is SL_{n+1}, the special linear group of $(n+1) \times (n+1)$ matrices with determinant one.

Type B_n (the odd-dimensional special orthogonal Lie algebra $\mathfrak{so}_{2n+1}$): For $n \geq 1$, let ω_o denote the $(2n+1) \times (2n+1)$ matrix with 1's along the anti-diagonal and 0's elsewhere. Then $\mathfrak{so}_{2n+1}$ consists of $x \in \mathfrak{sl}_{2n+1}$ such that $x^\top \omega_o + \omega_o x = 0$. (Using any other non-degenerate symmetric matrix instead of ω_0 gives an isomorphic Lie algebra.) The unique simply-connected algebraic group with Lie algebra $\mathfrak{so}_{2n+1}$ is called the *spin group*, and it is denoted by $Spin_{2n+1}$. We are going to study the spin group in more detail in later sections. We should mention that the Lie algebra of the familiar group SO_{2n+1} (the odd orthogonal group) is also $\mathfrak{so}_{2n+1}$, however, SO_{2n+1} is not simply-connected.

Type C_n (the symplectic Lie algebra $\mathfrak{sp}_{2n}$): Let s denote the skew-symmetric matrix

$$s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \text{where } I_n \text{ is the } n \times n \text{ identity matrix.}$$

(Using any other non-degenerate skew-symmetric matrix instead of s leads to an isomorphic Lie algebra.) Then $\mathfrak{sp}_{2n}$ is the space of all $x \in \mathfrak{sl}_{2n}$ such that $x^\top s + sx = 0$. The unique simply-connected algebraic group with Lie algebra $\mathfrak{sp}_{2n}$ is the *symplectic group* defined as $Sp_n = \{g \in GL_n : \ g^\top sg = s\}$.

Type D_n (the even-dimensional special orthogonal Lie algebra $\mathfrak{so}_{2n}$): Similar to the case of type B_n, let ω_e denote the $2n \times 2n$ matrix with 1's along the anti-diagonal and 0's elsewhere. (Using any other non-degenerate symmetric matrix gives an isomorphic Lie algebra.) The even-dimensional special orthogonal Lie algebra $\mathfrak{so}_{2n}$ consists of $x \in \mathfrak{sl}_{2n}$ satisfying $x^\top s + sx = 0$. The simply-connected group corresponding to $\mathfrak{so}_{2n}$ is another spin group, denoted by $Spin_{2n}$.

In addition to the above list of simple Lie algebras, there are five other "exceptional" types, labeled by G_2, F_4, E_6, E_7, and E_8. We are going to focus on G_2 after we gain more understanding of representation theory in the next several sections.

2.3. Roots, weights, and the generation of Lie algebras

An endomorphism $x : V \to V$ on a complex vector space is called *semi-simple* if it is diagonalizable. An element $x \in \mathfrak{g}$ is called semi-simple, if ad $x : \mathfrak{g} \to \mathfrak{g}$ is a semi-simple endomorphism. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called *toral*, if it consists of semi-simple elements only. A maximal toral subalgebra in a semi-simple Lie algebra is called a *Cartan subalgebra*.

Theorem 2.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra defined over $\mathbb{C}$ (or, more generally, over a field of characteristic 0). Then any two Cartan subalgebras in $\mathfrak{g}$ are conjugate to each other via an inner automorphism [1] of $\mathfrak{g}$.*

From now on we assume that $\mathfrak{g}$ is semi-simple. Let V be a $\mathfrak{g}$-module and let $\mathfrak{h}$ be a total subalgebra. Since $\mathfrak{h}$ is abelian its elements are simultaneously diagonalizable linear operators on V. (Recall: commuting endomorphism are simultaneously diagonalizable.) Thus, as an $\mathfrak{h}$-module, V decomposes into generalized eigenspaces: $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V(\lambda)$, where $V(\lambda) = \{v \in V : \ h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$. If nonempty, $V(\lambda)$ is called a *weight-space*, and the corresponding eigenvalue-functional $\lambda : \ \mathfrak{h} \to \mathbb{C}$ is called the *weight* of $V(\lambda)$. Note that there can only be finitely many weights for V is finite dimensional. We have an important special case when $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and the representation V is the adjoint action of $\mathfrak{g}$ on itself. In this case, the weight spaces are denoted by $\mathfrak{g}_\lambda$ instead of $V(\lambda) = \mathfrak{g}(\lambda)$. The *root space decomposition* is the corresponding weight space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda\in\Phi\subset\mathfrak{h}^*-0} \mathfrak{g}_\lambda, \tag{1}$$

where $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \ [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{h}\}$, $\lambda \in \Phi$.

The set of non-zero weights Φ of the adjoint representation has a fascinating combinatorial structure. Its elements are called the *roots* of $\mathfrak{g}$. We list some of the important

[1] Suppose $x \in \mathfrak{g}$ is such that $(\text{ad } x)^k = 0$ for some $k > 0$. Then $\exp \text{ad } x = \sum_{i=1}^k (\text{ad } x)^i/i!$ is a well defined automorphism of $\mathfrak{g}$. The group generated by all such automorphisms is called the *group of inner automorphisms* of $\mathfrak{g}$.

properties and introduce some terminology to be used later. None of the observations listed below is difficult to prove.

(1) The Cartan subalgebra $\mathfrak{h}$ is the weight-space $\mathfrak{g}_\zeta$ for the zero eigenvalue-functional that maps everything to 0.
(2) The set Φ spans $\mathfrak{h}^*$.
(3) If $\alpha \in \Phi$, then so is $-\alpha \in \Phi$.
(4) The weight spaces $\mathfrak{g}_\alpha$, $\alpha \in \Phi$ are all one-dimensional, and they are called *root subspaces.*
(5) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \Phi$. The equality holds if $\alpha + \beta$ is a root.
(6) For $\alpha \in \Phi$, let $T_\alpha \in \mathfrak{h}$ denote the dual of α with respect to the Killing form (hence, $\alpha(H') = B(T_\alpha, H')$ for all $H' \in \mathfrak{h}$) . Then for any $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, we have $[X, Y] = B(X, Y)T_\alpha$.
(7) Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and let $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ be such that $\alpha(H_\alpha) = 2$. (T_α and H_α are related to each other by $T_\alpha = 2H_\alpha/B(H_\alpha, H_\alpha)$.) Hence, the triplet X, Y, H_α generates a copy of $\mathfrak{sl}_2 \cong \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$ in $\mathfrak{g}$.
(8) For all $\alpha, \beta \in \Phi$, the value of the functional β on H_α is an integer, and furthermore,

$$\beta(H_\alpha) = 2B(H_\beta, H_\alpha)/B(H_\alpha, H_\alpha).$$

(9) If α and β are roots such that $\beta \neq \pm\alpha$, then there are largest positive integers p and r such that $\beta + i\alpha \in \Phi$ for all $-r \leq i \leq p$, and $\beta + j\alpha \notin \Phi$, if $j < -r$ or $j > p$. In fact, $r - p = \beta(H_\alpha)$.

Let R denote the *root lattice*, the free $\mathbb{Z}$-module in $\mathfrak{h}^*$ that is spanned by Φ. Denote by $E \subset \mathfrak{h}$ the real vector subspace spanned by the corresponding vectors $\{H_\alpha \in \mathfrak{h} : \alpha \in \Phi\}$. In particular, $\mathfrak{h} = E \otimes_\mathbb{R} \mathbb{C}$. It is not difficult to see now that the restriction of the Killing form $B|_{\mathfrak{h}\times\mathfrak{h}}$ to E is positive-definite. By dualizing we obtain an inner product on the dual vector space E^*. Let Ω_{H_α} denote the hyperplane $\Omega_{H_\alpha} = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0\}$. It is easy to see that the line spanned by α in $\mathfrak{h}^*$ is orthogonal to Ω_{H_α}, and this leads us to define an important discrete invariant.

The *Weyl group* $W = W(\mathfrak{g}, \mathfrak{h})$ is the finite group generated by the set of reflections $s_\alpha : E^* \to E^*$ with respect to hyperplanes Ω_{H_α}, $\alpha \in \Phi$ such that $s_\alpha(\alpha) = -\alpha$. It is easy to see that such a linear transformation is explicitly given by

$$s_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha, \qquad \beta \in E^*.$$

Notice, for $\beta \in \Phi$, $s_\alpha(\beta)$ is another root. In other words, W permutes the roots among themselves. Consider the lattice P in $\mathfrak{h}^*$ consisting of linear functionals $\beta \in \mathfrak{h}^*$ such that $\beta(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. We call P denote the *weight lattice.* It is clear that the root lattice R is a a sublattice of the weight lattice P. It is also clear that both R and P are W-invariant. Furthermore, since all Cartan subalgebras are conjugate, these lattices are invariants of a semi-simple Lie algebra, and W is independent of the choice of a Cartan subalgebra.

2.4. Abstract root systems

Now let E denote an arbitrary real, finite dimensional vector space equipped a positive-definite symmetric bilinear form (= inner product) $(\cdot,\cdot) : E \times E \to \mathbb{R}$. As before, for $\alpha \in E$, denote by $s_\alpha : E \to E$ the linear automorphism defined by reflecting with respect to the hyperplane Ω_α orthogonal to the line $\mathbb{R}\alpha$. A *root system* $\Phi \subset E$ is a finite collection of vectors such that

(1) $E = \mathrm{Span}_{\mathbb{R}}\Phi$,
(2) for $\alpha \in \Phi$, the only other scaler multiple of α that is contained in Φ is $-\alpha$,
(3) $s_\alpha\Phi = \Phi$ for every $\alpha \in \Phi$,
(4) ratios $2(\beta,\alpha)/(\alpha,\alpha) = 2\cos\theta\frac{||\beta||}{||\alpha||}$ for $\alpha, \beta \in \Phi$ are all integers. Here, θ is the angle between α and β.

For our Lie theory purposes, we take E as the dual of the real vector space spanned by $\{H_\alpha \in \mathfrak{h} : \ \alpha \in \Phi\}$, where Φ is the root system of the pair $(\mathfrak{g}, \mathfrak{h})$, and the inner product is the one that is induced from the Killing form.

We call Φ *irreducible*, if it is not a union of two other root systems. The root systems corresponding to the simple Lie algebras are in fact irreducible root systems. Because of the fourth axiom above, the ratios of the lengths of the roots and the angles between them cannot be arbitrary. Indeed, possible angles are easily seen to be

$$\pi/6, \ \ \pi/4, \ \ \pi/3, \ \ \pi/2, \ \ 2\pi/3, \ \ 3\pi/4, \ \ 5\pi/6.$$

This observation allows us to list all possible inequivalent, irreducible root systems.

The dimension of E is called the *rank* of the root system $\Phi \subset E$. In Figure 3, we list all possible rank one and rank two root systems. In Figure 4, after connecting each dot to the center of the corresponding polytope, we have all irreducible rank three root systems.

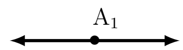
Remark 2.3. In a G_2 root system re-scaling short roots by multiplying each of them by 3, and keeping the lengths of the long roots the same results in another G_2 root system.

Let $\prec$ be a total ordering on E. (Total orders on a finite-dimensional vector space over an ordered field always exist.) Let $\Phi^+ \subset \Phi$ denote the set of roots which are "positive" with respect to this total ordering:

$$\alpha \in \Phi^+, \text{ if } 0 \prec \alpha.$$

Set of *simple roots* determined by Φ^+ is the maximal subset $\Delta \subseteq \Phi^+$ consisting of those positive roots which cannot be written as a sum of two other positive roots. Clearly, for each set of simple roots Δ, there is exactly one set of positive roots such that $\Delta \subseteq \Phi^+$.

Example 2.4. For the root system G_2 as depicted in Figure 1, $\{\alpha_1, \cdots, \alpha_6\}$ is a system of positive roots, and the set $\{\alpha_1, \alpha_2\}$ is a set of simple roots.



The unique (irreducible) rank 1 root system

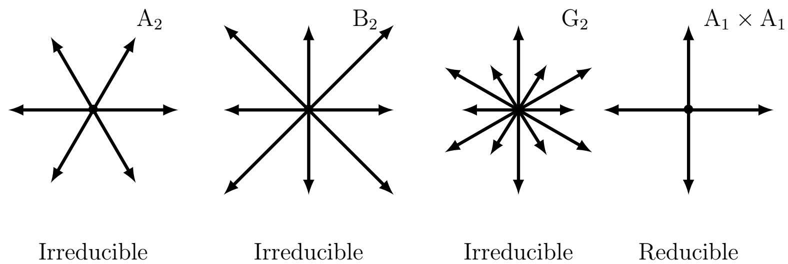


Figure 3. All rank one and rank two root systems

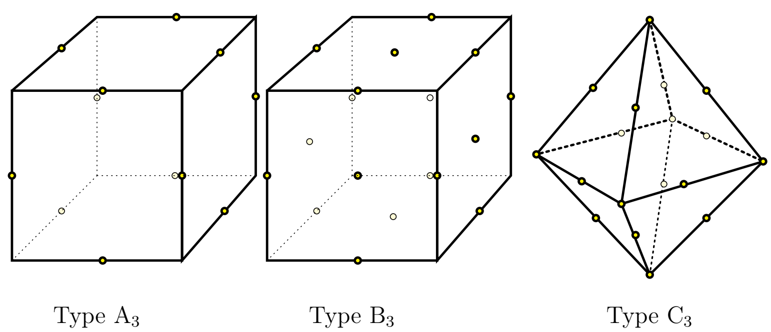


Figure 4. Irreducible roots systems of rank three.

Another characterization of simple roots is as follows:

(1) $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi$ is a basis for E.
(2) Any root α in Φ has an expansion of the form $\alpha = \sum_i n_i \alpha_i$ with either all n_i are non-negative integers, or all non-positive integers.

Finally, let us remark that, in Lie theory setting there is a convenient way to pick positive (hence, simple) roots. Suppose Φ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Choose a linear functional $f : \mathfrak{h}^* \to \mathbb{C}$ that takes irrational values on the weight lattice P. Then $\{\alpha \in \Phi : \ f(\alpha) > 0\}$ is a system of positive roots.

2.5. Serre's Theorem

Let $\mathcal{B}$ be a finite set. The *free Lie algebra on* $\mathcal{B}$ is the Lie algebra $\mathfrak{F}$ obtained from the free tensor algebra generated by $\mathcal{B}$ by declaring $A \otimes (B \otimes C) + B \otimes (C \otimes A) + C \otimes (A \otimes B) = 0$ and $A \otimes B = -B \otimes A$ for all $A, B, C \in \mathfrak{F}$. We denote the resulting multiplication by the bracket.

Theorem 2.3. *Let Φ be an irreducible abstract root system with a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, and let $\mathfrak{F}$ denote the free Lie algebra on the set of $3n$ variables $\mathcal{B} = \{H_1, \ldots, H_n, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$. For $1 \leq i, j \leq n$, define $n_{i,j}$ by setting $n_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$, hence $n_{ij} \in \{0, -1, -2, -3\}$ whenever $i \neq j$. Then the Lie algebra obtained from $\mathfrak{F}$ by further imposing the conditions below is a simple Lie algebra with root system Φ.*

(1) $[H_i, H_j] = 0$ *for all* i, j,
(2) $[X_i, Y_i] = H_i$ *for all* i, *and* $[X_i, Y_j] = 0$ *for all* $i \neq j$,
(3) $[H_i, X_j] = n_{ji} X_j$ *and* $[H_i, Y_j] = -n_{ji} Y_j$ *for all* i, j,
(4) *for all* $i \neq j$,
  (a) $[X_i, X_j] = [Y_i, Y_j] = 0$, *if* $n_{ji} = 0$;
  (b) $[X_i, [X_i, X_j]] = [Y_i, [Y_i, Y_j]] = 0$, *if* $n_{ji} = -1$;
  (c) $[X_i, [X_i, [X_i, X_j]]] = [Y_i, [Y_i, [Y_i, Y_j]]] = 0$, *if* $n_{ji} = -2$;
  (d) $[X_i, [X_i, [X_i, [X_i, X_j]]]] = [Y_i, [Y_i, [Y_i, [Y_i, Y_j]]]] = 0$, *if* $n_{ji} = -3$.

In the light of Serre's Theorem above, if we can identify a root system of a Lie algebra inside the root system of another, then we can explicitly determine how these two Lie algebras are interacting. It is very instructive to produce the root system of $\mathfrak{g}_2$ via that of $\mathfrak{so}_7$. To this end, let a, b and c be three non-adjacent corners of the cube containing all root vectors of $\mathfrak{so}_7$, see Figure 5. The six roots lying on the edges emanating from the corners a, b and c lie on a plane T that passes through the origin. Indeed, T is parallel to the abc-plane. There are exactly 6 other roots on the edges of the cube, 3 of which lie on one side of the plane T, and 3 on the other side. We project these roots orthogonally onto T. It is easy to verify that the resulting configuration of projected vectors forms a G_2 root system. (To visualize, place the cube by its corner on table in such a way that corners a, b and c have equal distance from the surface. Then project the root vectors on the table and observe the configuration of G_2. The parallel translation of the table surface to the center of the cube is T, and it contains all the root vectors for G_2.) In the same vein, the root system A_3 sits inside B_3. Indeed, we simply remove the root vectors that passes through the (centers of the) maximal faces of the cube. Finally, we see that A_2 lives on A_3 as the indicated in Figure 6.

In the sequel, we are going to use these embeddings of root systems to study the branchings of representations.

2.6. Representation Theory

Although we do not necessarily need it, for the sake of completeness we provide a construction of all irreducible representations of a semi-simple Lie algebra.

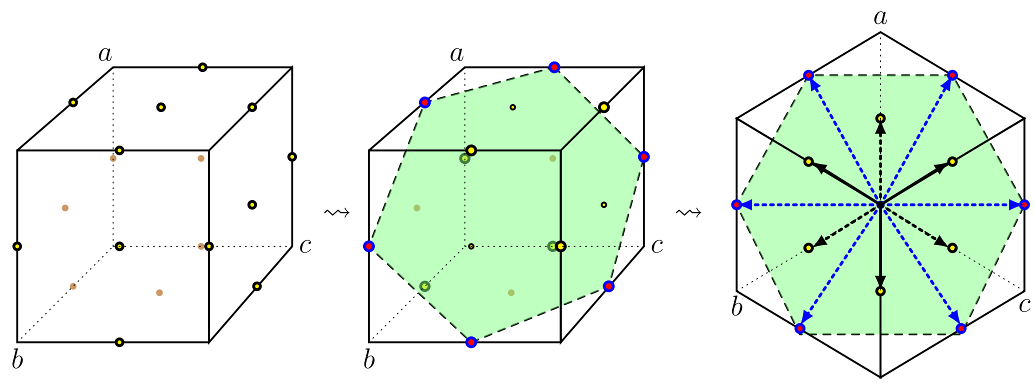


FIGURE 5. G_2 in B_3.

The universal enveloping algebra of a Lie group G is the algebra of left-invariant differential operators of all orders on G. A more concrete description can be obtained by passing to the Lie algebra of G. In fact, as we are going to see, the universal enveloping algebra is unavoidable when studying the representation theory of $\mathfrak{g}$.

Let $\mathrm{T}^i = \mathrm{T}^i(\mathfrak{g})$ denote the space of i-tensors on the underlying vector space of the Lie algebra $\mathfrak{g}$ (over some field $\mathbb{F}$), and denote by $\mathrm{T}(\mathfrak{g})$ the *tensor algebra* of $\mathfrak{g}$:

$$\mathrm{T}(\mathfrak{g}) = \bigoplus_{i \geq 0} \mathrm{T}^i(\mathfrak{g}).$$

We denote by $\mathrm{J}(\mathfrak{g}) \subset \mathrm{T}(\mathfrak{g})$ the two sided ideal that is generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y], \text{ for } x, y \in \mathfrak{g}.$$

The quotient algebra $\mathrm{U}(\mathfrak{g}) := \mathrm{T}(\mathfrak{g})/\mathrm{J}(\mathfrak{g})$ is the *universal enveloping algebra* of $\mathfrak{g}$, which is a Lie algebra itself. Since $\mathrm{J}(\mathfrak{g})$ is contained in $\bigoplus_{i>0} \mathrm{T}^i(\mathfrak{g})$ and since $\mathrm{T}^0(\mathfrak{g}) = \mathbb{F}$, a copy of $\mathbb{F}$ is contained in $\mathrm{U}(\mathfrak{g})$. More importantly, a copy of $\mathfrak{g}$ sits in $\mathrm{U}(\mathfrak{g})$. This is a part of important "Poincaré-Birkhoff-Witt Theorem" (abbreviated to PBW), which we explain next:

Define $\mathrm{U}^m = \mathrm{U}^m(\mathfrak{g}) \subset \mathrm{U}(\mathfrak{g})$ to be the image of $\mathrm{T}^0 \oplus \mathrm{T}^1 \oplus \cdots \oplus \mathrm{T}^m$ in $\mathrm{U}(\mathfrak{g})$. Notice that T^m is mapped onto $\mathrm{U}^m - \mathrm{U}^{m-1}$. For $m \geq 0$, we define the vector space $\mathrm{G}^m := \mathrm{U}^m/\mathrm{U}^{m-1}$. There is an obvious bilinear multiplication $\mathrm{G}^m \times \mathrm{G}^n \longrightarrow \mathrm{G}^{m+n}$. For notational consistency, set $\mathrm{U}^{-1} := 0$. Thus

$$\mathrm{Gr}(\mathfrak{g}) = \bigoplus_{m \geq 0} \mathrm{G}^m$$

is a graded, associative algebra with unity. It is rather long but explanatory to name $\mathrm{Gr}(\mathfrak{g})$ as *the associated graded algebra of the universal enveloping Lie algebra of* $\mathfrak{g}$.

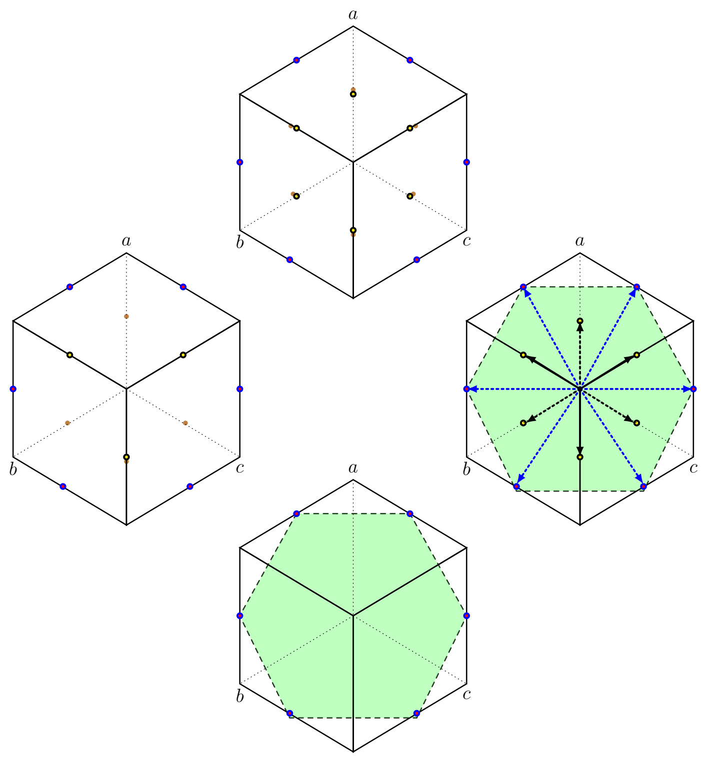


FIGURE 6. Embeddings via root systems

Let $\mathrm{Sym}^i(V)$ denote the vector space of symmetric i-tensors on a vector space V. Recall that the symmetric algebra $\mathrm{Sym}(V) = \bigoplus_{i \geq 0} \mathrm{Sym}^i(V)$ of V is naturally isomorphic to the $\mathbb{F}$-valued polynomial functions on V.

Theorem 2.4 (PBW)**.** *The associated graded ring of the universal enveloping algebra of* $\mathfrak{g}$ *is (graded-)isomorphic to* $\mathrm{Sym}(\mathfrak{g})$*. In other words,* $\mathrm{Gr}(\mathfrak{g})$ *is (graded-)isomorphic to the ring of polynomial functions on* $\mathfrak{g}$.

As a corollary of PBW we see that $\mathfrak{g}$ embeds into $\mathrm{U}(\mathfrak{g})$ as a Lie subalgebra. This observation is tremendously strengthened by the following following result.

Theorem 2.5. *Let* $X = \{x_1, x_2, \dots\}$ *be any ordered basis for* $\mathfrak{g}$*. Given a partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ *of* $m \in \mathbb{N}$ *(a non-increasing sequence of positive integers that sum to* m*), let* x_λ *denote the monomial* $x_\lambda = x_{\lambda_k} \otimes x_{\lambda_{k-1}} \otimes \cdots \otimes x_{\lambda_1} \in \mathrm{T}(\mathfrak{g})$*. Then as* λ *runs over all partitions of all positive integers, the set of images of* x_λ*'s in* $\mathrm{U}(\mathfrak{g})$ *constitutes a basis.*

Proof. Consider the subspace W of T^m generated by all monomials of the form x_λ, where λ is a partition of m. Then W is mapped onto Sym^m. It follows from PBW that the images of these monomials span G^m and are linearly independent. □

The role of universal enveloping algebras in the representation theory of $\mathfrak{g}$ begins with a consequence of its universal property.

Proposition 2.6. *For each Lie algebra homomorphism from* $\mathfrak{g}$ *into* $\mathfrak{gl}(V)$ *there is a unique algebra homomorphism* ρ_{U} *from* $\mathrm{U}(\mathfrak{g})$ *into* $\mathrm{U}(\mathfrak{gl}(V))$ *extending the Lie algebra homomorphism.*

It is easy to check that the converse of Proposition 2.6 is also true: Any algebra homomorphism out of $\mathrm{U}(\mathfrak{g})$ restricts to a Lie algebra homomorphism out of $\mathfrak{g}$.

2.7. Constructing the irreducible modules of $\mathfrak{g}$

Let $\mathfrak{g}_\alpha$ denote a root subspace as in (1) and let M be a $\mathfrak{g}$-module. Choosing a system of positive roots Φ^+ in $\Phi(\mathfrak{g},\mathfrak{h})$, we call an element $v^+ \in M$ *a highest weight vector*, if $\mathfrak{g}_\alpha \cdot v^+ = 0$ for all simple roots $\alpha \in \Delta$ associated with Φ^+. Treating M as a $\mathrm{U}(\mathfrak{g})$-module, we define a sub-representation V in M by setting $V = \mathrm{U}(\mathfrak{g}) \cdot v^+$. We call such a module *highest weight module* for $\mathfrak{g}$.

Theorem 2.7. *Let* $V = \mathrm{U}(\mathfrak{g}) \cdot v^+$ *be a highest weight module for* $\mathfrak{g}$*, hence in particular, a system of positive roots* Φ^+ *is chosen. Suppose that the elements of* Φ^+ *are ordered as in* $\beta_1, \ldots, \beta_m$*, and* $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ *is the set of simple roots in* Φ^+*. Suppose also that the maximal vector* v^+ *of* V *belongs to some weight space* $V(\lambda) \subset V$*. Then*

1. *As a vector space over* $\mathbb{F}$*,* V *is equal to*
$$V = \mathit{Span}_{\mathbb{F}}\{y_{\beta_1}^{i_1} y_{\beta_2}^{i_2} \cdots y_{\beta_m}^{i_m} : \ (i_1, \ldots, i_m) \in \mathbb{N}^m\}.$$
2. *The weights of* V *are of the form*
$$\mu = \lambda - \sum_{i=1}^{\ell} k_i \alpha_i, \quad \textit{for some } k_i \in \mathbb{N},\ i = 1, \ldots, \ell.$$
3. *For each* $\mu \in \mathfrak{h}^*$*, the weight subspace* $V(\mu)$ *is finite dimensional, and furthermore the "highest" weight space* $V(\lambda)$ *is one dimensional.*
4. V *is an indecomposable (irreducible)* $\mathfrak{g}$*-module with a unique maximal proper submodule and a corresponding unique irreducible quotient.*
5. *Every homomorphic image of* V *is also a highest weight module.*

It follows from the above theorem that a highest weight vector of a highest weight module V is unique up to a scalar multiple. What is slightly more complicated to prove is that two highest weight modules of the same *highest weight* λ have to be isomorphic. Next, we construct all highest weight modules explicitly.

A *Borel subalgebra* of $\mathfrak{g}$ is a subalgebra of the form $\mathfrak{b} = \mathfrak{h} \oplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. We denote by $\mathfrak{n}^-$ the subalgebra $\mathfrak{n}^- = \oplus_{\alpha\in-\Phi^+} \mathfrak{g}_\alpha$, so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$. Consequently, $\mathrm{U}(\mathfrak{g}) = \mathrm{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathrm{U}(\mathfrak{b})$.

For $\lambda \in \mathfrak{h}^*$, let C_λ denote a copy of $\mathbb{C}$ which is regarded as a $\mathfrak{b}$-module via

- $t \cdot v = \lambda(t)v$ for all $t \in \mathfrak{h}$,
- $\mathfrak{g}_\alpha \cdot v = v$ for all $\alpha \in \Phi^+$.

Now, we can define

$$M_\lambda := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} C_\lambda.$$

We call M_λ, the *Verma module* of highest weight λ. It is clear that M_λ is a highest weight module with highest weight vector $1 \otimes v^+$ of highest weight λ. In fact, M_λ is isomorphic to $\mathrm{U}(\mathfrak{g})/\mathrm{I}(\lambda)$, where $\mathrm{I}(\lambda)$ is the annihilator of the maximal vector v^+ in $\mathrm{U}(\mathfrak{g})$. Note that

$$M_\lambda = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} C_\lambda = (\mathrm{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathrm{U}(\mathfrak{b})) \otimes_{\mathrm{U}(\mathfrak{b})} C_\lambda \cong \mathrm{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} C_\lambda$$

as a vector space. We know from Theorem 2.7 that M_λ has an irreducible quotient, which we denote by V_λ. Indeed, V_λ is the quotient of M_λ by the sum of all proper submodules of M_λ. Conversely, any irreducible representation of $\mathfrak{g}$ with highest weight λ is such a quotient. Therefore the remaining question to answer is "when is an irreducible representation of $\mathfrak{g}$ finite dimensional ?"

Theorem 2.8. *An irreducible $\mathfrak{g}$-module V of highest weight λ is finite dimensional if and only if for each simple root $\alpha \in \Delta$ the ratio $\langle \lambda, \alpha \rangle := 2\dfrac{(\lambda,\alpha)}{(\alpha,\alpha)}$ is a nonnegative integer. Here, the pairing $(,)$ is the dual of the restriction of the Killing form on $\mathfrak{t} \times \mathfrak{t}$.*

A weight λ as in Theorem 2.8 is called *dominant* . The weight lattice P of $\mathfrak{g}$ is the free $\mathbb{Z}$-module consisting of $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Delta$. If Δ is indexed such that $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, then we define the set of "fundamental dominant weights" $\{\omega_1, \omega_2, \ldots, \omega_\ell\} \in \mathrm{P} \subset \mathfrak{t}^*$ as the dual basis of the "co-root basis" $\{H_{\alpha_1}, \ldots, H_{\alpha_\ell}\}$ for $\mathfrak{t}$. Therefore, $\omega_j(H_{\alpha_i}) = \delta_{i,j}$ for all $i, j = 1, \ldots, \ell$, or equivalently

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij}, \text{ for all } \alpha_j \in \Delta. \tag{2}$$

Example 2.5. We compute the fundamental weights for the root system G_2 which is depicted in Figure 1. A set of simple roots is given by $\Delta = \{\alpha_1, \alpha_2\}$. Reflecting α_6 with respect to the line whose normal is α_1 gives α_6 back. Thus, we have

$$\alpha_6 = s_{\alpha_1}(\alpha_6) = \alpha_6 - \alpha_6(H_{\alpha_1})\alpha_1$$

implying that $\alpha_6(H_{\alpha_1}) = 0$. Similarly, reflecting α_6 with respect to the line whose normal is α_2 gives α_5. Equivalently,

$$\alpha_5 = s_{\alpha_2}(\alpha_6) = \alpha_6 - \alpha_6(H_{\alpha_2})\alpha_2.$$

In order for this equality to be true, we must have $\alpha_6(H_{\alpha_2}) = 1$. Therefore, $\omega_1 = \alpha_6$. By the same procedure, we find that $\omega_2 = \alpha_4$.

The computation in Example 2.5 extends to the general situation as follows: Define $m_{i,j} := \langle \alpha_i, \alpha_j \rangle$ for $i, j = 1, \ldots, \ell$. Then writing $\alpha_i = \sum_{j=1}^{\ell} a_{i,j}\omega_j$, we see that $a_{i,j} = m_{i,j}$.

Of course, this implies $\omega_i = \sum_{j=1}^{\ell} m'_{i,j}\alpha_j$, where $(m'_{i,j})_{i,j=\ell}$ is the inverse of the matrix $(m_{i,j})_{i,j=\ell}$. In the literature, the latter matrix is known as the *Cartan matrix*.

It is evident that $\lambda \in P$ is dominant if and only if all coefficients in the expansion $\lambda = \sum k_i\omega_i$ are nonnegative. The *fundamental representations* of $\mathfrak{g}$ are those irreducible representations with the highest weights $\omega_1, \omega_2, \cdots \omega_3$. It is an important fact to remember that each weight is conjugate to a unique dominant weight via the action of Weyl group. Moreover, if λ is dominant, then $\sigma\lambda \preceq \lambda$ for all $\sigma \in W$. In fact, for a simple root α_i, by equation (2) we see that $s_{\alpha_i}(\omega_j) = \omega_j - \delta_{ij}\alpha_i$. Using this observation, the weights of an irreducible representation are computable by our next theorem:

Theorem 2.9. *Let $\lambda \in \mathrm{P}$ be a dominant weight with the corresponding irreducible representation $V = V_\lambda$. Suppose μ is a weight of V, α is a root, and i is an integer between 0 and $\langle \mu, \alpha \rangle$. Then $\lambda - i\alpha$ is also a weight of V. Moreoever, an arbitrary $\nu \in P$ is a weight of V if and only if ν and all of its W-conjugates are $\preceq \lambda$.*

The *multiplicity* of a weight space $V(\mu)$ in a representation V is the number of different copies of $V(\mu)$ appearing in the weight space decomposition of V. By Theorem 2.9, to understand the nature of V, it suffices to find the multiplicities of each weight $\mu \in \mathrm{P}$ such that $\mu \prec \lambda$. In fact, if $V = V_\lambda$ is irreducible, then the multiplicity of $V(\mu)$ in V is equal to

$$\sum_{w \in W} sign(w)\mathcal{P}(w(\lambda + \rho) - (\mu + \rho)),$$

where $\mathcal{P}$ is *Kostant's partition function* that we are going to introduce in Part II, and ρ is the half the sum of all positive roots and sign(w) equals the determinant of w as a linear map from $\mathfrak{h}^*$ to itself. This formula, known as *Kostant's multiplicity formula* is very similar in nature to Kostant's other famous formula, called "Kostant's branching formula" that we are going to utilize for studying branchings of representations in the next chapter.

3. Part II

3.1. Branching from SL_{n+1} to SL_n

The special linear group SL_{n+1} is simply connected, hence its representation theory is fully reflected by its Lie algebra $\mathfrak{sl}_{n+1}$. Its subset $\mathfrak{t} \subset \mathfrak{sl}_{n+1}$ consisting of all diagonal matrices forms a Cartan subalgebra. Let $\epsilon_i \in \mathfrak{t}^*$, $i = 1, \ldots, n+1$ denote the i-th coordinate function defined by $\epsilon_i(diag(a_1, \ldots, a_{n+1})) = a_i$. Then the set of roots associated with $(\mathfrak{sl}_{n+1}, \mathfrak{t})$ is $\Phi = \{\pm(\epsilon_i - \epsilon_j) : \ 1 \le i < j \le n+1\}$. For $i = 1, \ldots, n$, let α_i denote the root $\alpha_i = \epsilon_i - \epsilon_{i+1}$, and set $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Then Δ is a basis for the n-dimensional Euclidean space $E = \mathrm{Span}_{\mathbb{R}}\Phi$. It is easy to verify that any root $\alpha \in \Phi$ can be written in the form $\alpha = \sum_{\alpha_i \in \Delta} n_{\alpha_i}\alpha_i$, where all coefficients n_{α_i} are all non-positive, or all non-negative. Hence, Δ is a set of simple roots. The associated fundamental weights are easily seen to be equal to

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{n+1}(\epsilon_1 + \cdots + \epsilon_{n+1}) \tag{3}$$

for $i = 1, \ldots, n$.

Theorem 3.1 (Cartan)**.** *For $k = 1, \ldots, n$, the k-th exterior power $\bigwedge^k \mathbb{C}^{n+1}$ of the defining representation $\mathbb{C}^{n+1}$ is an irreducible representation of SL_{n+1} with the highest weight ω_k.*

Let G and H be two simply connected groups such that $H \subset G$. If V is a representation of G, then it is an H-module, as well. The determination of irreducible constituents of V as an H-module is called the *branching problem* of V. When viewed as an H-module, we denote V by $\mathrm{Res}_H^G(V)$. If μ is a dominant weight for H, we use the notation $m(V, \mu) \in \mathbb{Z}$ for the number of occurrences in V of the irreducible representation V_μ of H with the highest weight μ. Suppose G is SL_{n+1}, and let H denote the stabilizer subgroup in G of the standard basis element e_{n+1} of $\mathbb{C}^{n+1}$. Then H is isomorphic to SL_n.

Theorem 3.2 (Weyl)**.** *Let V_λ be an irreducible representation of G of highest weight λ. Then $m(V_\lambda, \mu) \leq 1$ for all dominant weights μ of H. Furthermore, $m(V_\lambda, \mu) = 1$ if and only if the coefficients of the expansions $\lambda = \sum_{i=1}^{n+1} \lambda_i \epsilon_i$ and $\mu = \sum_{i=1}^{n} \mu_i \epsilon_i$ satisfy the* interlacing *condition:*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_{n+1}. \tag{4}$$

Although the above theorem is very useful for computing branching multiplicities for arbitrarily large n, for smaller values of n we have alternative methods. Let us do this for $n = 3$ and for all fundamental weights of $\mathfrak{sl}_4$. The fundamental weights $\omega_1(\mathfrak{sl}_4), \omega_2(\mathfrak{sl}_4)$ and $\omega_3(\mathfrak{sl}_4)$ correspond to the irreducible representations $\mathbb{C}^4$, $\wedge^2\mathbb{C}^4$ and $\wedge^3\mathbb{C}^4$ in the given order. For SL_3, the correspondence is $\omega_1(\mathfrak{sl}_3) \leftrightsquigarrow \mathbb{C}^3$, and $\omega_2(\mathfrak{sl}_3) \leftrightsquigarrow \wedge^2\mathbb{C}^3$. Therefore,

(1) $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}$,

(2) $\wedge^2\mathbb{C}^4 = \wedge^2(\mathbb{C}^3 \oplus \mathbb{C}) = (\wedge^2\mathbb{C}^3 \otimes \wedge^0\mathbb{C}) \oplus (\wedge^1\mathbb{C}^3 \otimes \wedge^1\mathbb{C}) \oplus (\wedge^0\mathbb{C}^3 \otimes \wedge^2\mathbb{C})$
$= \wedge^2\mathbb{C}^3 \oplus \mathbb{C}^3$,

(3) $\wedge^3\mathbb{C}^4 = \wedge^3(\mathbb{C}^3 \oplus \mathbb{C})$
$= (\wedge^3\mathbb{C}^3 \otimes \wedge^0\mathbb{C}) \oplus (\wedge^2\mathbb{C}^3 \otimes \wedge^1\mathbb{C}) \oplus (\wedge^1\mathbb{C}^3 \otimes \wedge^2\mathbb{C}) \oplus (\wedge^0\mathbb{C}^3 \otimes \wedge^3\mathbb{C})$
$= \mathbb{C} \oplus \wedge^2\mathbb{C}^3$.

3.2. *Spin* group and its fundamental representations

Let V be a finite dimensional vector space over $\mathbb{C}$, and let $\mathrm{T}(V)$ be the tensor algebra of V. Suppose we have a symmetric, non-degenerate, bilinear form $D : V \times V \to \mathbb{C}$. We denote by $\mathrm{D}(V)$ the two sided ideal in $\mathrm{T}(V)$ that is generated by all elements of the form

$$x \otimes y + y \otimes x - D(x, y) \cdot 1_{\mathrm{T}(V)} \text{ for all } x, y \in V.$$

The quotient $\mathrm{Cliff}(V, D) := \mathrm{T}(V)/\mathrm{D}(V)$ is called the *Clifford algebra* of the pair (V, D), and furthermore, there is a natural $\mathbb{Z}/2$-grading on it. Therefore,

$$\mathrm{Cliff}(V, D) = \mathrm{Cliff}^+(V, D) \oplus \mathrm{Cliff}^-(V, D),$$

where $\mathrm{Cliff}^+(V, D)$ is the subalgebra generated by the products of even degree elements. Since it is an associative algebra, $\mathrm{Cliff}(V, D)$ has a natural Lie algebra structure defined by $[a, b] = a \cdot b - b \cdot a$. In fact, the identification of $\mathfrak{so}(V, D)$, which is the Lie algebra of endomorphisms of V that preserve D, with $\wedge^2 V$ gives a natural embedding $\mathfrak{so}(V, D) \hookrightarrow \mathrm{Cliff}^+(V, D)$ of Lie algebras.

Let $\gamma : V \hookrightarrow V \times V \to \mathrm{Cliff}(V, D)$ be the natural diagonal quotient map. The *pin group* on V is the subgroup of invertible elements of the Clifford algebra generated by the elements $-1_{\mathrm{Cliff}(V,D)}$ and $\gamma(v)$, $v \in V$ with $D(v, v) = -2$. Finally, the *spin group* $Spin(V, D)$ is defined as the identity component of $\mathrm{Pin}(V, D)$. As it is mentioned in the introduction, the spin group is simply-connected and there is a natural 2:1 covering homomorphism $\pi : Spin(V, D) \to SO(V, D)$, where $SO(V, D)$ is the group of invertible linear automorphisms of V which preserve D. Thus, the Lie algebra of $Spin(V, D)$ is the special orthogonal Lie algebra, consisting of endomorphisms of V that are skew-symmetric with respect to the bilinear form D.

There are important differences between spin groups of odd-dimensional vector spaces and the spin groups of even-dimensional spaces and this will be clear in a moment. When $\dim V = 2n + 1$, let $\mathcal{B} = \{e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}\}$ be an ordered basis of V such that

$$D(e_0, e_0) = 1 \text{ and } D(e_i, e_j) = \delta_{i,-j} \text{ for all } i, j \in \{-n, \dots, -1, 1, \dots, n\}.$$

Therefore, the matrix $S_{\mathcal{B}}$ of D in the basis $\mathcal{B}$ is equal to ω_o that is used to define $\mathfrak{so}_{2n+1}$. In this case, if $\pi : Spin(V, D) \to SO(V, D)$ is the double cover of $SO(V, D)$, then we let $H = \{g \in Spin(V, D) : \pi(g)e_0 = e_0\}$. It is easily seen that H is isomorphic to the spin group of $(V', D|_{V'})$, where $V' \subset V$ is the vector subspace that is spanned by $\mathcal{B}' = \{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$.

We use the basis $\mathcal{B}$ to identify the elements of $\mathfrak{so}(V, D)$ with matrices. The resulting Lie algebra is nothing but $\mathfrak{so}_{2n+1}$, and the Lie subalgebra corresponding to H in $\mathfrak{so}_{2n+1}$ is given by $\mathrm{Lie}(H) \cong \mathfrak{so}_{2n} = \{A \in \mathfrak{so}_{2n+1} : Ae_0 = 0\}$. We call $V = \mathbb{C}^{2n+1}$ the *defining representation* of $\mathfrak{so}_{2n+1}$. Similarly, the vector subspace that is spanned by $\mathcal{B}' = \{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$ is called the *defining representation* for $\mathfrak{so}_{2n}$.

The vector space $V = \mathbb{C}^{2n+1}$ has a decomposition $V \cong W \oplus \mathbb{C}e_0 \oplus W^*$, where $W \subset V$ is the span of first n basis vectors from $\mathcal{B}$ and W^* is the subspace spanned by the last n basis vectors. Let $\bigwedge^\bullet W$ (respectively $\bigwedge^\bullet W^*$) denote the exterior algebra on W (resp. on W^*). It turns out that the Clifford algebra $\mathrm{Cliff}(V, D)$ is isomorphic to the direct sum of matrix algebras:

$$\mathrm{Cliff}(V, D) \cong \mathrm{End}(\wedge^\bullet W) \oplus \mathrm{End}(\wedge^\bullet W^*).$$

Furthermore, the even part $\mathrm{Cliff}^+(V, D)$ is isomorphic to $\mathrm{End}(\wedge^\bullet W)$. In particular, this isomorphism gives a representation of $\mathfrak{so}_{2n+1}$ on $S := \wedge^\bullet W$, called the *spin-representation.*

In the case of an even-dimensional vector space $V = \mathbb{C}^{2n} \cong W \oplus W^*$ (W and W^* are as before) the Clifford algebra is isomorphic to a single copy of the matrix algebra:

$$\mathrm{Cliff}(V, D) \cong \mathrm{End}(\wedge^\bullet W),$$

and its even Lie subalgebra $\mathrm{Cliff}^+(V, D)$ is isomorphic to the direct sum of two Lie algebras

$$\mathrm{Cliff}^+(V, D) \cong \mathrm{End}(\oplus_i \wedge^{2i} W) \oplus \mathrm{End}(\oplus_i \wedge^{2i+1} W).$$

Thus $\mathfrak{so}(V, D)$ acquired two representations; one on $\oplus_i \wedge^{2i} W$, and one on $\oplus_i \wedge^{2i+1} W$. We call these representations the *half-spin representations*, and denote them by S^+ and S^-, respectively.

Diagonal matrices of the form $diag(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1)$, $a_i \in \mathbb{C}$ forms a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{so}_{2n+1}$. In fact, $\mathfrak{h}$ is the Lie algebra of the maximal torus

$$H = \{diag(x_1, \ldots, x_n, 1, x_n^{-1}, \ldots, x_1^{-1}) : \ x_i \in \mathbb{C}^*\} \subset SO_{2n+1}.$$

Let $\epsilon_i \in \mathfrak{h}^*$, $i = 1, \ldots, n$ denote the i-th coordinate function on diagonal matrices: $\epsilon_i(diag(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1)) = a_i$. It is clear that $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis for $\mathfrak{h}^*$. The associated roots of $(\mathfrak{so}_{2n+1}, \mathfrak{h})$ are

$$\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j) \ \text{ for } \ 1 \leq i < j \leq n \ \text{ together with } \ \pm e_i \ 1 \leq i \leq n.$$

Set $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_n = \epsilon_n$. Then $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a set of simple roots. The set of positive roots is then

$$\Phi^+ = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j : 1 \leq i < j \leq n\} \cup \{e_i : \ 1 \leq i \leq n\}.$$

Finally, the fundamental dominant weights are $\omega_i = \epsilon_1 + \cdots + \epsilon_i$ for $i = 1, \ldots, n-1$ and $\omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$.

For $\mathfrak{so}_{2n}$, the root datum is given as follows. We use the Cartan subalgebra

$$\mathfrak{h} = \{diag(a_1, \ldots, a_n, -a_n, \ldots, -a_1) : \ a_i \in \mathbb{C}\}.$$

As before, let ϵ_k denote the k-th coordinate function on $\mathfrak{h}$. Then

$$\Phi(\mathfrak{so}_{2n}, \mathfrak{h}) = \{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j) : \ 1 \leq i < j \leq n\}.$$

Set, as before, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$. Different from the odd case, let $\alpha_n = \epsilon_{n-1} + \epsilon_n$. The set of simple roots we use is $\Delta := \{\alpha_i : \ i = 1, \ldots, n\}$. Then the fundamental dominant weights are computed to be $\omega_i = \epsilon_1 + \cdots + \epsilon_i$ for $i = 1, \ldots, n-2$, and

$$\omega_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n) \ \text{ and } \ \omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n).$$

Theorem 3.3 (Cartan)**.** *For $k = 1, \ldots, n-1$, the k-th exterior power $\bigwedge^k \mathbb{C}^{2n+1}$ of the defining representation is an irreducible representation of $Spin_{2n+1}$ with highest weight ω_k. The irreducible representation corresponding to the fundamental weight ω_n is given by the spin-representation $S \cong \wedge^\bullet \mathbb{C}^n$.*

For the even spin group $Spin_{2n}$, the exterior powers $\bigwedge^k \mathbb{C}^{2n}$, $k = 1, \ldots, n-2$ are irreducible with the corresponding highest weights ω_k. The remaining two fundamental representations are the half-spin representations S^- and S^+ with the highest weights ω_{n-1} and ω_n, respectively.

3.3. Branching from $Spin_7$ to SL_3 via the right path

When expanded in the orthonormal basis $\epsilon_1, \dots, \epsilon_n$ of $\mathfrak{h}^*$, the dominant weights of $\mathfrak{so}_{2n+1}$ split into two classes, namely, 1) *integral weights*, which are of the form $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in \mathbb{Z}_+$ for all $i = 1, \dots, n$, and 2) *half-integral weights*, which are of the form $\lambda + \hbar_n$, where λ is an integral weight, and $\hbar_n = (\frac{1}{2}, \dots, \frac{1}{2})$. Note that the half-integral weights are those weights of $Spin_{2n+1}$ that are not representations of SO_{2n+1}.

Theorem 3.4 (Murnaghan)**.** *Let V_λ be an irreducible representation of $Spin_{2l+1}$. The branching from $Spin_{2l+1}$ to $H \cong Spin_{2l}$ is multiplicity free. Furthermore, $m(\lambda, \mu) = 1$ if and only if the following two conditions are satisfied:*

(1) *either both μ and λ are integral weights, or they both are half-integral weights.*
(2) *they interlace: $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n \geq |\mu_n|$.*

Once again, initially, we are going to avoid using the above high-tech theorem for branching computations. We look at the case of $\mathfrak{g} = \mathfrak{so}_7$, which has three fundamental representations:

(1) $\omega_1(\mathfrak{so}_7)$ corresponding to its defining representation $\mathbb{C}^7$.
(2) The adjoint representation $\wedge^2\mathbb{C}^7$ with the highest weight $\omega_2(\mathfrak{so}_7)$.
(3) The spin-representation $S \cong \wedge^\bullet\mathbb{C}^3$ with the highest weight $\omega_3(\mathfrak{so}_7)$.

We begin with the defining representation $\mathbb{C}^7$. Obviously,

$$\mathbb{C}^7 = \mathbb{C}^6 \oplus \mathbb{C},$$

which is equal to the sum of defining representation for $\mathfrak{so}_6$ with the one dimensional trivial representation.

In the case of $\mathfrak{so}_7$ the adjoint representation is the second exterior power of the defining representation, which is one of the fundamental representations. However, this is not the case for $\mathfrak{so}_6$. Indeed, the highest weight of the second exterior power $\wedge^2\mathbb{C}^6$ is given by $\epsilon_1 + \epsilon_2 = \omega_2(\mathfrak{so}_6) + \omega_3(\mathfrak{so}_6)$.

The adjoint representation of $\mathfrak{so}_7$ decomposes into irreducibles of $\mathfrak{so}_6$ as follows:

$$\begin{aligned}\wedge^2\mathbb{C}^7 \;=\; \wedge^2(\mathbb{C}^6 \oplus \mathbb{C}) \;&=\; (\wedge^0\mathbb{C}^6 \otimes \wedge^2\mathbb{C}) \oplus (\wedge^1\mathbb{C}^6 \otimes \wedge^1\mathbb{C}) \oplus (\wedge^2\mathbb{C}^6 \otimes \wedge^0\mathbb{C})\\ &=\; \mathbb{C}^6 \oplus \wedge^2\mathbb{C}^6\end{aligned}$$

the sum of defining representation and the adjoint representation. Finally, the spin representation $S = \wedge^*\mathbb{C}^3$ splits into

$$S = \mathbb{C} \oplus \mathbb{C}^3 \oplus \wedge^2\mathbb{C}^3 \oplus \wedge^3\mathbb{C}^3 \;=\; (\mathbb{C}^3 \oplus \wedge^3\mathbb{C}^3) \oplus (\mathbb{C} \oplus \wedge^2\mathbb{C}^3) \;=\; S^- \oplus S^+.$$

Obviously, this is true in general: the spin representation of $\mathfrak{so}_{2n+1}$ splits into a sum of half-spin representations of $\mathfrak{so}_{2n}$.

Remark 3.1. There are important coincidences among the small rank simple Lie algebras and one of these coincidences is between $\mathfrak{so}_6$ and $\mathfrak{sl}_4$. Indeed, these two simple Lie algebras are the same. This can be seen from their root systems both of which are of type A_3.

However, as far as the notation for weights are concerned we must be careful. Recall that the fundamental weights for SL_4 correspond to the irreducible representations $\mathbb{C}^4$, $\wedge^2\mathbb{C}^4$ and $\wedge^3\mathbb{C}^4$ in the given order. From this we see that $\omega_2(\mathfrak{so}_6)$ is $\omega_1(\mathfrak{sl}_4)$, $\omega_3(\mathfrak{so}_6)$ is $\omega_3(\mathfrak{sl}_4)$, and $\omega_1(\mathfrak{so}_6)$ is $\omega_2(\mathfrak{sl}_4)$.

Next, we compute the branching of an irreducible representation V of $\mathfrak{so}_7$ of highest weight $\lambda = a\omega_1(\mathfrak{so}_7) + b\omega_2(\mathfrak{so}_7) + c\omega_3(\mathfrak{so}_7)$ to $\mathfrak{sl}_3$. In terms of ϵ-coordinates,

$$\lambda = a(1,0,0) + b(1,1,0) + c(1/2,1/2,1/2) = (a+b+c/2, b+c/2, c/2).$$

If $c \neq 0$, then according to the Branching Theorem an irreducible representation of $\mathfrak{so}_6$ with the highest weight $\mu = (\mu_1, \mu_2, \mu_3)$ is a constituent of the restriction if and only if μ is a half-integer weight satisfying

$$a+b+c/2 \;\geq\; \mu_1 \;\geq\; b+c/2 \;\geq\; \mu_2 \;\geq\; c/2 \;\geq\; |\mu_3|$$

Equivalently,

$$\begin{aligned} 2a+2b+c \;&\geq\; 2\mu_1 \;\geq\; 2b+c, \\ a+b \;&\geq\; \mu_1-\mu_2 \;\geq\; 0, \\ b+c \;&\geq\; \mu_2-\mu_3 \;\geq\; 0. \end{aligned}$$

Note that $\omega_1(\mathfrak{so}_6) = \epsilon_1$, $\omega_2(\mathfrak{so}_6) = 1/2(\epsilon_1+\epsilon_2-\epsilon_3)$ and $\omega_3(\mathfrak{so}_6) = 1/2(\epsilon_1+\epsilon_2+\epsilon_3)$. Therefore,

$$\begin{aligned} \epsilon_1 &= \omega_1(\mathfrak{so}_6), \\ \epsilon_2 &= \omega_2(\mathfrak{so}_6) + \omega_3(\mathfrak{so}_6) - \omega_1(\mathfrak{so}_6), \\ \epsilon_3 &= \omega_3(\mathfrak{so}_6) - \omega_2(\mathfrak{so}_6), \end{aligned}$$

implying

$$\mu \;=\; \mu_1\epsilon_1+\mu_2\epsilon_2+\mu_3\epsilon_3 \;=\; (\mu_1-\mu_2)\omega_1(\mathfrak{so}_6) + (\mu_2-\mu_3)\omega_2(\mathfrak{so}_6) + (\mu_2+\mu_3)\omega_3(\mathfrak{so}_6).$$

On the other hand, by Remark 3.1, in terms of the fundamental weights of $\mathfrak{sl}_4$ this is equal to

$$\mu = (\mu_2-\mu_3)\omega_1(\mathfrak{sl}_4) + (\mu_1-\mu_2)\omega_2(\mathfrak{sl}_4) + (\mu_2+\mu_3)\omega_3(\mathfrak{sl}_4). \tag{5}$$

In view of (3), the expression (5) for μ in terms of the coordinate functions $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ of the Cartan subalgebra of $\mathfrak{sl}_4$ is equal to

$$\frac{(\mu_1+\mu_2+\mu_3)}{2}\epsilon_1 + \frac{(\mu_1-\mu_2+\mu_3)}{2}\epsilon_2 + \frac{(-\mu_1+\mu_2+\mu_3)}{2}\epsilon_3 - \frac{(\mu_1+\mu_2+\mu_3)}{2}\epsilon_4.$$

Now, by using Theorem 3.2 we see that an irreducible representation $U = U_\tau$ of $\mathfrak{sl}_3$ appears in the decomposition of the irreducible representation corresponding to μ if and only if its highest weight $\tau = \tau_1\epsilon_1 + \tau_2\epsilon_2 + \tau_3\epsilon_3$ satisfies

$$\frac{(\mu_1+\mu_2+\mu_3)}{2} \geq \tau_1 \geq \frac{(\mu_1-\mu_2+\mu_3)}{2} \geq \\ \tau_2 \geq \frac{(-\mu_1+\mu_2+\mu_3)}{2} \geq \tau_3 \geq -\frac{(\mu_1+\mu_2+\mu_3)}{2}.$$

Summarizing, we obtain the following result.

Theorem 3.5 (Branching from $Spin_7$ to SL_3)**.** *An irreducible representation U of $\mathfrak{sl}_3$ with highest weight $\tau = d\omega_1(\mathfrak{sl}_3) + e\omega_2(\mathfrak{sl}_3)$ appears in a decomposition of the irreducible representation V of $\mathfrak{so}_7$ of highest weight $\lambda = a\omega_1(\mathfrak{so}_7) + b\omega_2(\mathfrak{so}_7) + c\omega_3(\mathfrak{so}_7)$ if and only if exactly one of the following holds:*

(1) *λ is a half-integral weight (that is to say $c \neq 0$), and there exists a half-integral weight $\mu = (\mu_1, \mu_2, \mu_3) \in (1/2, 1/2, 1/2) + \mathbb{Z}^3_{\geq 0}$ of $\mathfrak{so}_6$ such that $a+b+c/2 \geq \mu_1 \geq b+c/2 \geq \mu_2 \geq c/2 \geq |\mu_3|$, and*

$$\frac{(\mu_1+\mu_2+\mu_3)}{2} \geq \frac{2d+e}{3} \geq \frac{(\mu_1-\mu_2+\mu_3)}{2} \geq \frac{-d+e}{3} \geq \\ \frac{(-\mu_1+\mu_2+\mu_3)}{2} \geq \frac{-(d+2e)}{3} \geq -\frac{(\mu_1+\mu_2+\mu_3)}{2}.$$

(2) *λ is an integral weight (that is to say $c = 0$), and there exists an integral weight $\mu = (\mu_1, \mu_2, 0)$ of $\mathfrak{so}_6$ such that $a+b \geq \mu_1 \geq b \geq \mu_2 \geq 0$, and*

$$\frac{(\mu_1+\mu_2)}{2} \geq \frac{2d+e}{3} \geq \frac{(\mu_1-\mu_2)}{2} \geq \frac{-d+e}{3} \geq \\ \frac{(-\mu_1+\mu_2)}{2} \geq \frac{-(d+2e)}{3} \geq -\frac{(\mu_1+\mu_2)}{2}.$$

3.4. Kostant's Partition Function

A *rational character* of a linear algebraic group is a group homomorphism $\chi : G \to \mathbb{C}^*$ which is at the same time a morphism of algebraic sets. The set of rational characters $X(G)$ forms an abeliean group with respect to point-wise multiplication. Note that characters are class functions, namely, they take constant value on conjugacy classes. Recall that in a semi-simple Lie algebra all Cartan subalgebras are conjugate. Using the "exponential map" $\exp : \mathfrak{g} \to G$ we see that all maximal tori in G are conjugate. On the other hand, it is known that the union of all maximal tori in a connected, semi-simple algebraic group is a dense subset. Therefore, a character χ of a connected, semi-simple algebraic group G is completely determined by its values on a maximal torus. In other words, $X(G) = X(T)$. If, in addition, G is simply-connected, then the character group $X(T)$ is isomorphic to the weight-lattice of $(\mathfrak{g}, \mathfrak{t})$.

We fix a Borel subgroup B of G containing T and denote by Φ, Φ^+ and Δ, respectively, the root system, the subset of positive roots, and the set of simple roots, relative to (G, B, T).

Let V be a finite dimensional $\mathfrak{g}$-module. If G is simply-connected, then there exists a representation $\pi : G \to GL(V)$ whose differential is the given representation of $\mathfrak{g}$. All finite dimensional representations of G are obtained in this way. As a finite dimensional $\mathfrak{g}$-module, the vector space V decomposes into a direct sum of $\mathfrak{t}$ weight spaces $V(\mu)$ with $\mu \in \mathrm{P}$. Assuming V is irreducible, let $P(V) \subset \mathrm{P}$ denote the set of weights that appear in the weight space decomposition of V. Then there exists a unique dominant weight $\lambda \in P(V)$ such that $\mu \prec \lambda$ for all $\mu \in P(V)$.

The *formal character* $\mathrm{ch}(V)$ of a representation $\pi : G \to GL(V)$ is defined to be

$$\mathrm{ch}(V) = \sum_{\mu \in P(V)} d_\mu e^\mu \in \mathbb{Z}[\mathrm{P}], \quad \text{where} \quad d_\mu = \dim V(\mu).$$

Here, we use the notation $e^\mu, \mu \in \mathrm{P}$ to pass to the multiplicative notation. Indeed, the sum $\mu + \nu$ of any weights $\mu, \nu \in \mathrm{P}$ is conveniently represented by the corresponding multiplication $e^\mu \cdot e^\nu = e^{\mu+\nu}$. If ω_i is a fundamental weight and $\pi_i : G \to GL(V_{\omega_i})$ is the associated irreducible representation of G, then e^{ω_i} is the formal character corresponding to

$$\mathrm{ch}(V_{\omega_i})(t) = \mathrm{Tr}(\pi_i(t)).$$

In other words, we view a formal character $\mathrm{ch}(V)$ as a regular function on the maximal torus T via $\mathrm{ch}(V)(t) = \mathrm{Tr}(\pi(t))$, $t \in T$.

After this reminder on the characters, we look at the behavior of the restriction of a formal character of G to a maximal torus of a subgroup $H \subseteq G$. We are going to use "Kostant's Branching Multiplicity" formula for our analysis. To this end, let T_G and $T_H = H \cap T_G$ denote the maximal tori for G and H, respectively, and let $\mathfrak{t}_\mathfrak{g}$ and $\mathfrak{t}_\mathfrak{h}$ denote the corresponding Lie algebras. We choose a set of positive roots $\Phi_\mathfrak{g}^+$ for $(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$. Following Goodman and Wallach [GoWa] (page 370), we make the following regularity assumption[2]:

Regularity Assumption: Suppose there exists an element $X_0 \in \mathfrak{t}_\mathfrak{h}$ such that

$$\langle \alpha, X_0 \rangle > 0 \text{ for all } \alpha \in \Phi_\mathfrak{g}^+.$$

This is equivalent to finding a regular element $X_0 \in \mathfrak{t}_\mathfrak{h}$ (not touching the "walls of the Weyl chamber"), which is regular in $\mathfrak{t}_\mathfrak{g}$, also. In particular, the vector X_0 allows us to define a total ordering on the roots of $(\mathfrak{h}, \mathfrak{t}_\mathfrak{h})$:

$$\Phi_\mathfrak{h}^+ = \{\gamma \in \Phi_\mathfrak{h} : \ \langle \gamma, X_0 \rangle > 0\}.$$

For a weight λ of $\mathfrak{g}$, let $\overline{\lambda}$ denote the restriction of λ to the subalgebra $\mathfrak{h}$. Set $\overline{\Phi_\mathfrak{g}^+} := \{\overline{\alpha} : \ \alpha \in \Phi_\mathfrak{g}^+\}$, and for $\beta \in \overline{\Phi_\mathfrak{g}^+}$ denote by R_β the set of all $\alpha \in \Phi_\mathfrak{g}^+$ such that $\overline{\alpha} = \beta$. Let Σ_0 denote the set of positive roots β from $\Phi_\mathfrak{h}^+$ with $|R_\beta| > 1$, and let

[2]This assumption is not necessary, as D. Vogan removes it in [Vog]. However, it simplifies the proof and it is helpful for choosing an appropriate positive root system. In effect, it boils down to the following algebraic fact: the centralizer in G of the maximal torus T_H is commutative, hence it is equal to T_G.

$\Sigma_1 := \overline{\Phi_{\mathfrak{g}}^+} - \Phi_{\mathfrak{h}}^+$. The union $\Sigma_0 \cup \Sigma_1$ is denoted by Σ and the multiplicity of an element $\beta \in \Sigma$ is defined by

$$m_\beta = \begin{cases} |R_\beta| & \text{if } \beta \notin \Phi_{\mathfrak{h}}^+, \\ |R_\beta| - 1 & \text{if } \beta \in \Phi_{\mathfrak{h}}^+. \end{cases}$$

The *Kostant partition function* $\mathcal{P} = \mathcal{P}_\Sigma$ on the dual Cartan subalgebra $\mathfrak{t}_{\mathfrak{h}}^*$ is determined by

$$\prod_{\beta \in \Sigma} (1 - e^{-\beta})^{-m_\beta} = \sum_\eta \mathcal{P}(\eta) e^{-\eta}.$$

Observe that $\mathcal{P}(\eta)$ is equal to the number of ways of writing η as non-negative integral combinations of the elements of Σ.

Theorem 3.6 (Kostant's Branching Multiplicity Formula)**.** *Let λ and μ be dominant integral weights for G and H, respectively, and let V_λ and V_μ denote the corresponding irreducible representations. Then*

$$m(\lambda, \mu) = \sum_{s \in W_{\mathfrak{g}}} sign(s) \mathcal{P}(\overline{s \cdot (\lambda + \rho_{\mathfrak{g}})} - \mu - \overline{\rho_{\mathfrak{g}}}).$$

3.5. Branching from G_2 to SL_3

In a type G_2 root system, the set of all long-roots forms an A_2 root system $\Phi_{\mathfrak{h}} = \{\alpha_2, \alpha_5, \alpha_6, -\alpha_2, -\alpha_5, -\alpha_6\}$, see Figure 1. As in the introduction section we have the following associated list of generating variables

$$\mathcal{B}' = \{H_2, H_5, H_6, X_2, Y_2, X_5, Y_5, X_5, Y_6\}.$$

Since these variables are among the generating set of variables for $\mathfrak{g}_2$, Serre's relations 1 through 4 are automatically satisfied, and hence, we have a copy of $\mathfrak{sl}_3$ naturally sitting inside $\mathfrak{g}_2$. Since $H_5 = H_1 + H_2$ and $H_6 = H_1 + 2H_2$ (see Table 1), the Cartan subalgebra $\mathfrak{t} = Span_{\mathbb{C}}\{H_2, H_5\}$ of $\mathfrak{sl}_3$ is equal to that of $\mathfrak{g}_2$. Therefore, the regularity assumption is also automatically satisfied, and hence we can apply Kostant's formula for computing the branching multiplicities from G_2 to SL_3.

Let $X_0 = 3H_2 + H_5$, and set $\Phi_{\mathfrak{g}_2}^+ = \{\alpha_i : \ i = 1, \ldots, 6\}$ as usual. Then $\langle \alpha, X_0 \rangle > 0$ for all $\alpha \in \Phi_{\mathfrak{g}_2}^+$, and moreover, $\{\alpha \in \Phi_{\mathfrak{sl}_3} : \ \langle \alpha, X_0 \rangle > 0\} = \{\alpha_2, \alpha_5, \alpha_6\}$ is a set of positive roots for $(\mathfrak{sl}_3, \mathfrak{t})$. Since $[H_i, X_j] = \alpha_j(H_i) X_j$ for all i and j, by using Table 1, the values of the positive roots on H_i's are easily computed as in Table 2.
In particular, we see that $|R_\beta| = 1$ for all $\beta \in \overline{\Phi_{\mathfrak{g}_2}^+}$, and therefore $\Sigma = \Sigma_1 = \{\overline{\alpha_1}, \overline{\alpha_3}, \overline{\alpha_4}\}$. Thus, the Kostant partition function $\mathcal{P}$ of $\mathfrak{t}^*$ is obtained from the generating function equality

$$\frac{1}{1 - e^{-\alpha_1}} \frac{1}{1 - e^{-\alpha_3}} \frac{1}{1 - e^{-\alpha_4}} = \sum_{\eta \in \mathfrak{t}^*} \mathcal{P}(\eta) e^{-\eta}. \tag{6}$$

$\alpha_1(H_2) = -1$	$\alpha_1(H_5) = \alpha_1(H_1) + \alpha_1(H_2) = 2 - 1 = 1$
$\alpha_2(H_2) = 2$	$\alpha_2(H_5) = \alpha_2(H_1) + \alpha_2(H_2) = -3 + 2 = -1$
$\alpha_3(H_2) = 1$	$\alpha_3(H_5) = \alpha_3(H_1) + \alpha_3(H_2) = -1 + 1 = 0$
$\alpha_4(H_2) = 0$	$\alpha_3(H_5) = \alpha_4(H_1) + \alpha_4(H_2) = 1 + 0 = 1$
$\alpha_5(H_2) = -1$	$\alpha_5(H_5) = \alpha_5(H_1) + \alpha_5(H_2) = 3 - 1 = 2$
$\alpha_6(H_2) = 1$	$\alpha_6(H_5) = \alpha_6(H_1) + \alpha_6(H_2) = 0 + 1 = 1$

TABLE 2. Values of the positive roots

Lemma 3.7. *Suppose $\mu = a\alpha_1 + b\alpha_2 \in \mathfrak{t}^*$ for some non-negative integers a and b. Then*

$$\mathcal{P}(a\alpha_1 + b\alpha_2) = \begin{cases} \min\{a-b, b\} + 1 & \text{if } a \geq b \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First of all, observe that, since $\mathcal{P}(a\alpha_1 + b\alpha_2)$ is equal to number of ways of writing μ as a non-negative integral combinations of α_1, $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$ we must have that $a \geq b \geq 0$. Expanding the left hand side of the equation (6), we see that $\mathcal{P}(\eta)$ is equal to number of $(i, j, k) \in \mathbb{Z}^3_{\geq 0}$ such that $a = i + j + 2k$ and $b = j + k$. It follows that $a - b = i + k$, and therefore, each choice of k uniquely determines both j and i. Clearly, k is from $\{0, \ldots, \min\{b, a-b\}\}$, hence the proof is complete. $\square$

Let $\Delta = \{\alpha_1, \alpha_2\}$. The Weyl group $W_{\mathfrak{g}_2}$ is generated by the reflections $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$, and it has 12 elements. The action of these elements on Δ are listed in Table 3.

$s \in W_{\mathfrak{g}_2}$	α_1	α_2	$(n+5)\alpha_1 + (m+3)\alpha_2$
id	α_1	α_2	$(n+5)\alpha_1 + (m+3)\alpha_2$
s_1	$-\alpha_1$	$3\alpha_1 + \alpha_2$	$(-n+3m+4)\alpha_1 + (m+3)\alpha_2$
s_2	$\alpha_1 + \alpha_2$	$-\alpha_2$	$(n+5)\alpha_1 + (n-m+2)\alpha_2$
s_1s_2	$2\alpha_1 + \alpha_2$	$-3\alpha_1 - \alpha_2$	$(2n-3m+1)\alpha_1 + (n-m+2)\alpha_2$
s_2s_1	$-\alpha_1 - \alpha_2$	$3\alpha_1 + 2\alpha_2$	$(-n+3m+4)\alpha_1 + (-n+2m+1)\alpha_2$
$s_1s_2s_1$	$-2\alpha_1 - \alpha_2$	$3\alpha_1 + 2\alpha_2$	$(-2n+3m-1)\alpha_1 + (-n+2m+1)\alpha_2$
$s_2s_1s_2$	$2\alpha_1 + \alpha_2$	$-3\alpha_1 - 2\alpha_2$	$(2n-3m+1)\alpha_1 + (n-2m-1)\alpha_2$
$s_1s_2s_1s_2$	$\alpha_1 + \alpha_2$	$-3\alpha_1 - 2\alpha_2$	$(n-3m-4)\alpha_1 + (n-2m-1)\alpha_2$
$s_2s_1s_2s_1$	$-2\alpha_1 - \alpha_2$	$3\alpha_1 + \alpha_2$	$(-2n+3m-1)\alpha_1 + (-n+m-2)\alpha_2$
$s_1s_2s_1s_2s_1$	$-\alpha_1 - \alpha_2$	α_2	$(-n-5)\alpha_1 + (-n+m-2)\alpha_2$
$s_2s_1s_2s_1s_2$	α_1	$-3\alpha_1 - \alpha_2$	$(n-3m-4)\alpha_1 + (-m-3)\alpha_2$
$s_1s_2s_1s_2s_1s_2$	$-\alpha_1$	$-\alpha_2$	$(-n-5)\alpha_1 + (-m-3)\alpha_2$

TABLE 3. The action of $W_{\mathfrak{g}_2}$ on Δ

By using Table 3 we compute the values $\mathcal{P}(\overline{s\cdot(\lambda+\rho_{\mathfrak{g}})}-\mu-\overline{\rho_{\mathfrak{g}}})$. To this end, let $\lambda = n'\omega_1 + m'\omega_2$, $n', m' \geq 0$ be a highest weight for an irreducible representation V_λ of $\mathfrak{g}_2$. Since $\omega_1 = \alpha_6 = 3\alpha_1 + 2\alpha_2$ and $\omega_2 = \alpha_4 = 2\alpha_1 + \alpha_2$ (by Example 2.5) we see that $\lambda = (3n' + 2m')\alpha_1 + (2n' + m')\alpha_2$. Set $n = 3n' + 2m'$ and set $m = 2n' + m'$. Then we have $2m \geq n > m > 0$ unless V_λ is the trivial representation, in which case $n = m = n' = m' = 0$. Finally, note that $\rho_{\mathfrak{g}_2} = 5\alpha_1 + 3\alpha_2$.

The weight lattice of $\mathfrak{sl}_3$ ($\subset \mathfrak{g}_2$) is spanned by the fundamental dominant weights dual to H_{α_2} and H_{α_5}. Thus, $\omega_1 = \frac{1}{3}(2\alpha_5 + \alpha_2) = 2\alpha_1 + \alpha_2$ and $\omega_2 = \frac{1}{3}(\alpha_5 + 2\alpha_2) = \alpha_1 + \alpha_2$. If $\mu = a'\omega_1 + b'\omega_2$ is the highest weight of an irreducible epresentation of $\mathfrak{sl}_3$, then we express it in terms of α_1 and α_2's as $\mu = (2a'+b')\alpha_1 + (a'+b')\alpha_2$. Letting $a = 2a'+b'$ and $b = a' + b'$ we see that $a \geq b \geq 0$ with $a = 0$ if and only if V_μ is the trivial representation of $\mathfrak{sl}_3$.

With the above notation ($\lambda = n\alpha_1 + m\alpha_2$ with $n > m > 0$ and $\mu = a\alpha_1 + b\alpha_2$ with $a \geq b \geq 0$) we list the values of the Kostant partition function in Table 4.

$s \in W_{\mathfrak{g}_2}$	$\overline{s\cdot(\lambda+\rho_{\mathfrak{g}})}-\mu-\overline{\rho_{\mathfrak{g}}}$	$sign(s)\mathcal{P}(\overline{s\cdot(\lambda+\rho_{\mathfrak{g}})}-\mu-\overline{\rho_{\mathfrak{g}}})$
id	$(n-a)\alpha_1 + (m-b)\alpha_2$	$\min\{n-a-m+b, m-b\}+1$
s_1	$(-n+3m-a-1)\alpha_1 + (m-b)\alpha_2$	$-\min\{-n+2m-a-b-1, m-b\}-1$
s_2	$(n-a)\alpha_1 + (n-m-b-1)\alpha_2$	$-\min\{m-a+b+1, n-m-b-1\}-1$
s_1s_2	$(2n-3m-a-4)\alpha_1 + (n-m-b-1)\alpha_2$	0
s_2s_1	$(-n+3m-a-1)\alpha_1 + (-n+2m-b-2)\alpha_2$	0
$s_1s_2s_1$	$(-2n+3m-a-6)\alpha_1 + (-n+2m-b-2)\alpha_2$	0
$s_2s_1s_2$	$(2n-3m-a-4)\alpha_1 + (n-2m-b-4)\alpha_2$	0
$s_1s_2s_1s_2$	$(n-3m-a-9)\alpha_1 + (n-2m-b-4)\alpha_2$	0
$s_2s_1s_2s_1$	$(-2n+3m-a-6)\alpha_1 + (-n+m-b-5)\alpha_2$	0
$s_1s_2s_1s_2s_1$	$(-n-a-10)\alpha_1 + (-n+m-b-5)\alpha_2$	0
$s_2s_1s_2s_1s_2$	$(n-3m-a-9)\alpha_1 + (-m-b-6)\alpha_2$	0
$s_1s_2s_1s_2s_1s_2$	$(-n-a-10)\alpha_1 + (-m-b-6)\alpha_2$	0

TABLE 4. Computing multiplicities

Remark 3.2. In Table 4 we have 0's in the last column for the reason that either the coefficient of α_i is zero, or the coefficient of α_1 is less than that of α_2.

We conclude from Table 4 that

$$\begin{aligned} m(\lambda,\mu) = {} & \mathcal{P}((3n'+2m'-2a'-b')\alpha_1 + (2n'+m'-a'-b')\alpha_2) \\ & - \mathcal{P}(3n'+m'-2a'-b'-1)\alpha_1 + (2n'+m'-a'-b')\alpha_2) \\ & - \mathcal{P}((3n'+2m'-2a'-b')\alpha_1 + (n'+m'-a'-b'-1)\alpha_2) \end{aligned}$$

For convenience let us abbreviate this summation in the form $A - B - C$, where A is the first summand, B is the second, and C is the third summand. In particular, A, B and C are all non-negative. Let us denote the coefficient of α_1 in A by u_1, the coefficient of α_2 by v_1. Similarly, we define u_2, v_2, u_3 and v_3 to be the coefficients of α_i's in the second and the third summands.

Lemma 3.8. *If $m(\lambda, \mu) > 0$, then*

(1) $u_1 = 3n' + 2m' - 2a' - b' > 0$,
(2) $v_1 = 2n' + m' - a' - b' \geq 0$,
(3) $u_1 - v_1 = n' + m' - a' \geq 0$.

Proof. For $A-B-C$ to be positive, A has to be positive. The rest follows from Lemma 3.7. □

We analyze $A - B - C$ by looking at A more closely. First, assume that $u_1 - v_1 > v_1$, or equivalently that $b' \geq n'$. Hence $A = \min\{u_1 - v_1, v_1\} + 1 = v_1 + 1$. In this case, since $u_3 = u_1$ and $v_3 = v_1 - n' - 1$, we have $C = \min\{u_3 - v_3, v_3\} + 1 = v_1 - n' \geq 0$. Similarly, since $v_2 = v_1$ and $u_2 = u_1 - m' - 1$, we have $B = \min\{u_1 - v_1 - m' - 1, v_1\} + 1$. However, if $u_2 - v_2 \geq v_2$, then $B = v_1 + 1$ forcing $A - B - C \leq 0$. Therefore, we must have $B = u_1 - v_1 - m' - 1$ and

$$m(\lambda, \mu) = A - B - C = a' + 2, \quad \text{whenever } b' \geq n'.$$

Also, we have obtained the inequalities

$$n' < a' + b' \leq 2n' + m' \text{ and } a', b' \leq n' + m'.$$

Next we assume that $0 \leq u_1 - v_1 < v_1$, equivalently $0 \leq b' < n'$. Thus $A = u_1 - v_1 + 1$, and $B = u_1 - v_1 - m' \geq 0$. Since

$$\begin{aligned} C &= \min\{u_3 - v_3, v_3\} + 1 = \min\{2n' + m' - a' - 1', n' + m' - a' - b' - 1\} + 1 \\ &= n' + m' - a' - b' \geq 0, \end{aligned}$$

we have

$$m(\lambda, \mu) = -n' + a' + b' + 1 > 0 \text{ whenever } b' < n'.$$

In particular, as before, we have

$$2n' + m' \geq a' + b' \geq n' \text{ and } n' + m' \geq a' + b' \geq a', b'.$$

Conversely, if $(a', b') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ satisfies $2n' + m' \geq a' + b' \geq n'$ and $a', b' \leq n' + m'$, then $3n' + 2m' - a' - b' \geq 0$. It follows that $m(\lambda, \mu) \neq 0$ by analyzing the cases $b' \geq n'$ and $b' < n'$ just as we did above. We summarize these observations as follows:

Theorem 3.9. *Let $\lambda = n'\omega_1(\mathfrak{g}_2) + m'\omega_2(\mathfrak{g}_2)$ and $\mu = a'\omega_1(\mathfrak{sl}_3) + b'\omega_2(\mathfrak{sl}_3)$ be the highest weights of the irreducible representations of $\mathfrak{g}_2$ and $\mathfrak{sl}_3$, respectively. Then $m(\lambda, \mu) \neq 0$ if and only if*

(1) $n' \leq a' + b' \leq 2n' + m'$, *and*
(2) $a', b' \leq n' + m'$.

Furthermore, in this case,

$$m(\lambda,\mu) = \begin{cases} -n' + a' + b' + 1 & \textit{whenever } b' < n', \\ a' + 2 & \textit{whenever } b' \geq n'. \end{cases}$$

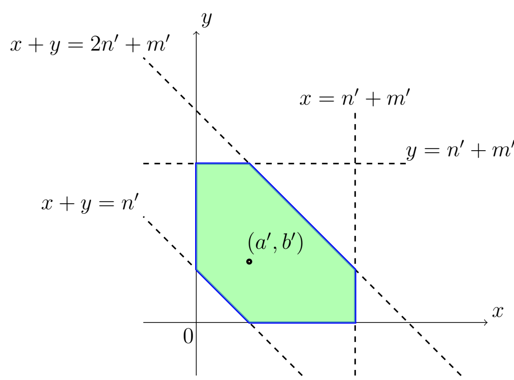


FIGURE 7. Branching region for (n', m')

3.6. Branching from $Spin_7$ to G_2

We use the embedding of $\mathfrak{g}_2$ into $\mathfrak{so}_7$ resulting from the corresponding root system embeddings. Let H_i denote the matrix $E_{i,i} - E_{-i,-i}$ for $i = 1, \ldots, 3$. Then $\{H_1, H_2, H_3\}$ forms a basis for a Cartan subalgebra $\mathfrak{t}_{\mathfrak{so}_7}$ of $\mathfrak{so}_7$. Let η_i denote the functional deftermined by $\eta_i(H_j) = 2\delta_{i,j}$. Then

$$\Phi_{\mathrm{B}_3} := \{\pm\eta_1, \pm\eta_2, \pm\eta_3\} \cup \{\pm(\eta_i + \eta_j), \pm(\eta_i - \eta_j) :\ 1 \leq i < j \leq 3\}$$

forms the root system for the pair $(\mathfrak{so}_7, \mathfrak{t}_{\mathfrak{so}_7})$. We use

$$\Delta_{\mathrm{B}_3} = \{\alpha_1 = \eta_1 - \eta_2,\ \alpha_2 = \eta_2 - \eta_3,\ \alpha_3 = \eta_3\}$$

as a set of simple roots for Φ_{B_3}, and hence, we get $\Phi^+_{\mathrm{B}_3}$ is

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, 2\alpha_2 + \alpha_3\},$$

which equals

$$\{\eta_i \pm \eta_j :\ 1 \leq i < j \leq 3\} \cup \{\eta_1, \eta_2, \eta_3\},$$

and is the associated set of positive roots.

Let us verify that there is a regular element X_0 in $\mathfrak{t}_{\mathfrak{so}_7}$. Let $h_1 = \frac{1}{3}H_1 - \frac{1}{3}H_2 + \frac{2}{3}H_3$ and $h_2 = H_2 - H_3$, and set $X_0 = 5h_1 + 3h_2 = \frac{5}{3}H_1 + \frac{4}{3}H_2 + \frac{1}{3}H_3$. Since $\alpha_i(X_0) > 0$ for $i = 1, 2, 3$, we see that for any positive root $\beta \in \Phi^+_{\mathrm{B}_3}$, $\beta(X_0) > 0$, hence X_0 is the desired element.

Let $\mathfrak{t}^*_{\mathfrak{g}_2} \subseteq \mathfrak{t}^*_{\mathfrak{so}_7}$ be the subspace defined by the equation $-\eta_1 + \eta_2 + \eta_3 = 0$. We depict $\mathfrak{t}^*_{\mathfrak{g}_2}$ as in Figure 8. Thus, the long roots of G_2 correspond to the solid dots, and they are given by $-\eta_1 - \eta_2, -\eta_2 + \eta_3, \eta_1 + \eta_3, \eta_1 + \eta_2, \eta_2 - \eta_3, -\eta_1 - \eta_3$.

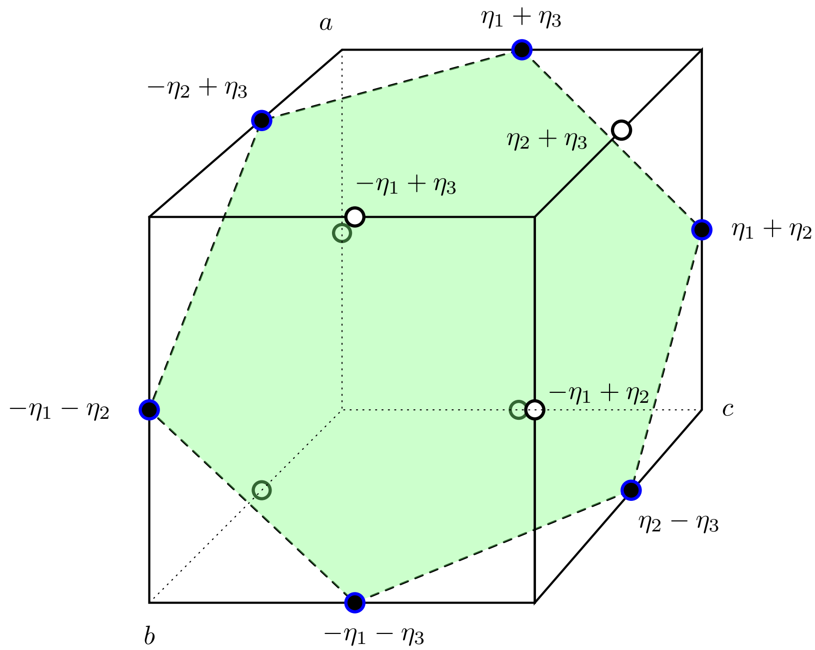


FIGURE 8. G_2 in B_3

The projections of the other six roots $\pm(\eta_1 - \eta_3), \pm(-\eta_1 + \eta_2), \pm(\eta_2 + \eta_3) \in \mathrm{B}_3$ onto $\mathfrak{t}^*_{\mathfrak{g}_2}$-plane are: $\pm\frac{1}{3}(-\eta_1 - 2\eta_2 + \eta_3)$, $\pm\frac{1}{3}(-\eta_1 + \eta_2 - 2\eta_3)$, and $\pm\frac{1}{3}(2\eta_1 + \eta_2 + \eta_3)$, respectively. These 12 vectors form a root system of type G_2. Since $\Phi^+_{\mathrm{G}_2} = \{\gamma \in \Phi_{\mathrm{G}_2} : \langle \gamma, X_0 \rangle > 0\}$, the following is a set of positive roots for G_2:

$$\{\eta_1 + \eta_2, \eta_1 + \eta_3, \eta_2 - \eta_3, \frac{1}{3}(\eta_1 + 2\eta_2 - \eta_3), \frac{1}{3}(\eta_1 - \eta_2 + 2\eta_3), \frac{1}{3}(2\eta_1 + \eta_2 + \eta_3)\}.$$

Let γ_1 denote $\frac{1}{3}(\eta_1 - \eta_2 + 2\eta_3)$ and let γ_2 denote $\eta_2 - \eta_3$. Then the ratio of the lengths $|\gamma_2|/|\gamma_1|$ is $\sqrt{3}$, and the angle between γ_1 and γ_2 is $5\pi/6$. Therefore $\Delta_{\mathrm{G}_2} := \{\gamma_1, \gamma_2\}$ is the set of simple roots with γ_1 being the short root, and γ_2 the long root.

The restrictions of the positive roots of Φ_{B_3} are

$$\begin{aligned}
\overline{\eta_3} = \overline{\eta_1 - \eta_2} &= \frac{1}{3}(\eta_1 - \eta_2 + 2\eta_3),\\
\overline{\eta_2} = \overline{\eta_1 - \eta_3} &= \frac{1}{3}(\eta_1 + 2\eta_2 - \eta_3),\\
\overline{\eta_1} = \overline{\eta_2 + \eta_3} &= \frac{1}{3}(2\eta_1 + \eta_2 + \eta_3),\\
\overline{\eta_1 + \eta_2} &= \eta_1 + \eta_2,\\
\overline{\eta_1 + \eta_3} &= \eta_1 + \eta_3,\\
\overline{\eta_2 - \eta_3} &= \eta_2 - \eta_3.
\end{aligned}$$

It follows that $\overline{\Phi^+_{B_3}} = \Phi^+_{G_2}$, and that Σ_0 consists of short roots of G_2 each of which has multiplicity 1. Therefore, we see that $\Sigma = \Sigma_0 = \{\gamma_1, \gamma_1 + \gamma_2, 2\gamma_1 + \gamma_2\}$.

If $\mu \in \mathfrak{t}^*_{\mathfrak{g}_2}$, then, by definition, $\mathcal{P}(\mu)$ is the number of ways of writing μ as in

$$\mu = a\gamma_1 + b(\gamma_1 + \gamma_2) + c(2\gamma_1 + \gamma_2) \quad \text{for some } a, b, c \in \mathbb{N}.$$

In particular, if $\mathcal{P}(\mu)$ is non-zero, then μ has to be of the form $a'\gamma_1 + b'\gamma_2$ for some $a', b' \in \mathbb{N}$ with $a' \geq b'$. Moreover, Lemma 3.7 is applicable, and hence,

$$\mathcal{P}(a'\gamma_1 + b'\gamma_2) = \begin{cases} \min\{a' - b', b'\} + 1 & \text{if } a' \geq b' \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

For any subset $A \subset \mathbb{Z}$, we set the notation $\delta_{A,i}$ for

$$\delta_{A,i} = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

Let $W_{\mathfrak{so}_7}$ denote the Weyl group of $\mathfrak{so}_7$. The standard permutation representation of $W_{\mathfrak{so}_7}$ on $\mathfrak{t}_1^*$ provides us with a useful description of its elements: Each $\sigma' \in W_{\mathfrak{so}_7}$ is of the form $\sigma' = \varepsilon_A \sigma$, where A is a subset of $\{1, 2, 3\}$, σ is a permutation of the indices of the coordinates η_1, η_2, η_3, and $\varepsilon_A \cdot \eta_i = (-1)^{\delta_{A,i}} \eta_i$.

Now, let $\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3$ be a dominant weight for $\mathfrak{so}_7$, hence $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ are either all non-negative integers, or all non-negative half-integers. For $\sigma' = \varepsilon_A\sigma$ in $W_{\mathfrak{so}_7}$, let $I(\sigma'; \lambda)$ denote $\sigma' \cdot (\lambda - \rho) - \rho = a\eta_1 + b\eta_2 + c\eta_3$, where $\rho = \rho_{\mathfrak{so}_7} = \frac{1}{2}(5\eta_1 + 3\eta_2 + \eta_3)$. For an arbitrary permutation σ of $\{1, 2, 3\}$, we express ρ in the form

$$\rho = \frac{7 - 2\sigma_1}{2}\eta_{\sigma_1} + \frac{7 - 2\sigma_2}{2}\eta_{\sigma_2} + \frac{7 - 2\sigma_3}{2}\eta_{\sigma_3}.$$

Hence, $I(\varepsilon_A\sigma;\lambda) = a\eta_1 + b\eta_2 + c\eta_3$ can be concisely written as

$$\begin{aligned} I(\varepsilon_A\sigma;\lambda) &= \sum_{i=1}^{3}\left((-1)^{\delta_{A,\sigma_i}}\lambda_i - \frac{(7-2i)(-1)^{\delta_{A,\sigma_i}} + (7-2\sigma_i)}{2}\right)\eta_{\sigma_i} \\ &= \sum_{i=1}^{3}\left((-1)^{\delta_{A,\sigma_i}}\lambda_i - \frac{7(1+(-1)^{\delta_{A,\sigma_i}}) - 2((-1)^{\delta_{A,\sigma_i}}i + \sigma_i)}{2}\right).\eta_{\sigma_i} \qquad (8) \end{aligned}$$

On the other hand, a straightforward calculation shows that the projection of a vector $a\eta_1 + b\eta_2 + c\eta_3$ onto the $-\eta_1 + \eta_2 + \eta_3 = 0$ plane is

$$\frac{2a+b+c}{3}\eta_1 + \frac{a+2b-c}{3}\eta_2 + \frac{a-b+2c}{3}\eta_3 = (2a+b+c)\gamma_1 + (a+b)\gamma_2.$$

Suppose $\mu = s\gamma_1 + r\gamma_2$ is a dominant weight for $\mathfrak{g}_2$, hence $s, r \in \mathbb{N}$ with $s - r > 0$. (This is because $\mu = s'\omega_1 + r'\omega_2 = s'(2\gamma_1+\gamma_2) + r'(3\gamma_1 + 2\gamma_2) = (2s'+3r')\gamma_1 + (s'+2r')\gamma_2$ for some $s', r' \geq 0$ with $s' + r' > 0$.) Therefore,

$$\mathcal{P}(\overline{I(\varepsilon_A\sigma;\lambda)} - \mu) = \mathcal{P}((2a+b+c-s)\gamma_1 + (a+b-r)\gamma_2),$$

which is equal to

$$\begin{cases} \min\{a+c-(s-r), a+b-r\} + 1 & \text{if } \ 2a+b+c-s \geq a+b-r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that in order for $2a+b+c-s \geq a+b-r \geq 0$ to hold, both inequalities $a + c \geq (s-r) > 0$ and $a + b \geq r > 0$ have to hold. Consequently, we have some restrictions on the subsets $A \subseteq \{1,2,3\}$ in order for $\mathcal{P}(\overline{I(\varepsilon_A\sigma;\lambda)} - \mu)$ to have a non-zero value. Indeed, if a, b, and c denotes the coordinates of (8), and if U_i, $i = 1,2,3$ denotes

$$U_i = -\frac{7(1+(-1)^{\delta_{A,\sigma_i}}) - 2((-1)^{\delta_{A,\sigma_i}}i + \sigma_i)}{2}, \qquad (9)$$

then

$$a + b = (-1)^{\delta_{A,\sigma_i}}\lambda_i + (-1)^{\delta_{A,\sigma_j}}\lambda_j + U_i + U_j, \qquad (10)$$

$$a + c = (-1)^{\delta_{A,\sigma_i}}\lambda_i + (-1)^{\delta_{A,\sigma_k}}\lambda_k + U_i + U_k, \qquad (11)$$

where $\sigma_i = 1, \sigma_j = 2$ and $\sigma_k = 3$.

By using (9) – (10), we analyze the (im)possibilities:

Case of $A = \{1,2,3\}$:

If $i = 1, j = 2$, then $U_i = 0, U_j = 0$, hence $a + b < 0$. If $i = 1, j = 3$, then $U_i = 0, U_j = -1$, hence $a + b < 0$. Therefore, $i = 1$ implies that $a + b < 0$.

If $i = 2, j = 3$, then $U_i = -1, U_j = -1$, hence $a + b < 0$. If $i = 2, j = 1$, then $U_i = -1, U_j = 1$, hence $a + b < 0$.

If $i = 3, j = 1$, then $U_i = -2, U_j = 1$, hence $a + b < 0$. If $i = 3, j = 2$, then $U_i = -2, U_j = 0$, hence $a + b < 0$.

Since in all of the above cases $a+b<0$, we conclude that when $A=\{1,2,3\}$, the value of the Kostant partition function is zero.

Case of $A=\{1,2\}$:

If $i=1, j=2$, then $U_i=0, U_j=0$, hence $a+b<0$. If $i=1, j=3$, then $U_i=0, U_j=-1$, hence $a+b<0$.

If $i=2, j=1$, then $U_i=0, U_j=1$ then $a+b$ may not be <0. However, in this case, $k=3$, and $U_3=0$, hence $a+c<0$.

If $i=3, j=1$, then $U_i=-2, U_3=1$ then $a+b<0$. If $i=3, j=2$, then $U_i=-2, U_3=0$ then $a+b<0$.

Since in all of the above cases either $a+b<0$, or $a+c<0$, we see that when $A=\{1,2\}$ the value of the Kostant partition function is zero.

Case of $A=\{1,3\}$:

If $i=1, j=2$, then $U_i=0, U_j=-3$, hence $a+b<0$. If $i=1, j=3$, then $U_i=0, U_j=-2$, hence $a+b<0$.

If $i=2, j=1$, then $U_i=-1, U_j=-4$, hence $a+b<0$. If $i=2, j=3$, then $U_i=-1, U_j=-2$, hence $a+b<0$.

If $i=3, j=1$, then $U_i=-1, U_j=-4$ then $a+b<0$. If $i=3, j=2$, then $U_i=-1, U_j=-3$ then $a+b<0$.

Since in all of the above cases either $a+b<0$, we see that when $A=\{1,3\}$ the value of the Kostant partition function is zero.

Case of $A=\{1\}$:

If $i=1, j=2$, then $U_i=0, U_j=-3$, hence $a+b<0$. If $i=1, j=3$, then $U_i=0, U_j=-2$, hence $a+b<0$.

If $i=2, j=1$, then $U_i=-1, U_j=-4$, hence $a+b<0$. If $i=2, j=3$, then $U_i=-1, U_j=-2$, hence $a+b<0$.

If $i=3, j=1$, then $U_i=-1, U_j=-4$ then $a+b<0$. If $i=3, j=2$, then $U_i=-1, U_j=-3$ then $a+b<0$.

Since in all of the above cases either $a+b<0$, we see that when $A=\{1\}$ the value of the Kostant partition function is zero.

Case of $A=\{2,3\}$:

In this case, we look at $i>1$ and $i=1$ separately.

If $i=2, j=1$, then $U_i=-4, U_j=1$, hence $a+b<0$. If $i=2, j=3$, then $k=1$, and $U_i=-4, U_k=-1$, hence $a+c<0$.

If $i=3, j=1$, then $U_i=-3, U_j=1$ then $a+b<0$. If $i=3, j=2$, then $U_i=-3, U_j=0$ then $a+b<0$.

We conclude that, if $A=\{2,3\}$ and $i>1$, then either $a+b<0$, or $a+c<0$, hence the value of the Kostant partition function is zero.

On the other hand, if $i = 1$, then the cases are $j = 2, k = 3$ and $j = 3, k = 2$. In the former case, $a+b = \lambda_1 - \lambda_2 - 5$, $a+c = \lambda_1 - \lambda_3 - 5$. In the latter case, $a+b = \lambda_1 - \lambda_2 - 4$, $a + c = \lambda_1 - \lambda_3 - 6$. The value of the partition function is easily computable for these two cases. We are going to summarize the end result of these computations (and of the other similar cases) in our final theorem below.

Case of $A = \{2\}$:

If $i = 2, j = 1$, then $U_i = -4, U_j = 1$, hence $a + b = -\lambda_2 + \lambda_1 - 3$. However, in this case, $k = 3$, and $a + c = -\lambda_2 + \lambda_3 - 4$, which is negative. Similarly, if $i = 2, j = 3$, then $k = 1$, and $U_i = -4, U_k = -1$. Hence $a + c = -\lambda_2 + \lambda_3 - 3 < 0$.

If $i = 3, j = 1$, then $U_i = -3, U_j = 1$, then $a + b = \lambda_3 - \lambda_1 - 2$, which is negative. If $i = 3, j = 2$, then $U_i = -3, U_j = 0$, then $a + b < 0$.

Thus, we see from the above cases that if $i > 1$, then the corresponding value of the partition function is zero. Therefore, we assume $i = 1$. Once again, there are two cases: $j = 2, k = 3$, and $j = 3, k = 2$. In the former case, $a+b = \lambda_1 - \lambda_2 - 5$, $a+c = \lambda_1 + \lambda_3 - 6$, and in the latter case $a + b = \lambda_1 - \lambda_3 - 6$, $a + c = \lambda_1 + \lambda_2 - 4$.

Case of $A = \{3\}$:

If $i = 2, j = 1$, then $U_i = -4, U_j = -4$, hence $a + b = -\lambda_2 + \lambda_1 - 8$. However, in this case, $k = 3$, and $a + c = -\lambda_2 + \lambda_3$, which is negative. Similarly, if $i = 2, j = 3$, then $k = 1$, and $U_i = -4, U_k = -1$. Hence $a + c = -\lambda_2 + \lambda_3 - 3 < 0$.

If $i = 3, j = 1$, then $U_i = -3, U_j = 1$, then $a + b = \lambda_3 - \lambda_1$, which is negative. If $i = 3, j = 2$, then $U_i = -3, U_j = 0$, then $a + b < 0$.

Thus, we see from the above cases that if $i > 1$, then the value of the Kostant partition function is zero. Therefore, we assume $i = 1$. Once again, there are two cases: $j = 2$, $k = 3$, and $j = 3$, $k = 2$. In the former case, $a + b = \lambda_1 - \lambda_2 - 5$, $a + c = \lambda_1 + \lambda_3 - 6$. In the latter case, $a + b = \lambda_1 - \lambda_3 - 6$, $a + c = \lambda_1 + \lambda_2 - 4$.

Case of $A = \emptyset$:

In this case, in both sums $a + b$ and $a + c$ the coefficients of λ_i's are positive, and $U_i = i + \sigma_i$ for $i = 1, 2, 3$.

Given two integers u and v, let us denote by $p(u, v)$ the function

$$p(u,v) = \begin{cases} \min\{u,v\} + 1 & \text{if both } u \text{ and } v \text{ are non-negative,} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

We conclude from the above analysis that

Theorem 3.10. *Let η_1, η_2, η_3 denote the dual of the standard coordinate functions on $\mathfrak{t}_{\mathfrak{so}_7}$, a Cartan subalgebra of $\mathfrak{so}_7$, and let $\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3 \in \mathfrak{t}^*_{\mathfrak{so}_7}$ be the highest weight of an irreducible $Spin_7$-representation V_λ. Then the multiplicity of an irreducible* *representation V_μ of $\mathfrak{g}_2$ with the highest weight $\mu = s'\omega_1(\mathfrak{g}_1) + r'\omega_2(\mathfrak{g}_2)$ in V_λ is given by*

$$\begin{aligned} m(\lambda,\mu) = {} & p(\lambda_1 - \lambda_2 - 5 - s' - r', \lambda_1 - \lambda_3 - 5 - s' - 2r') \\ & - p(\lambda_1 - \lambda_2 - 4 - s' - r', \lambda_1 - \lambda_3 - 6 - s' - 2r') \\ & - 2p(\lambda_1 - \lambda_2 - 5 - s' - r', \lambda_1 + \lambda_3 - 6 - r' - 2s') \\ & + 2p(\lambda_1 - \lambda_3 - 6 - s' - r', \lambda_1 + \lambda_2 - 4 - s' - 2r') \\ & + p(\lambda_1 + \lambda_2 + 6 - s' - r', \lambda_1 + \lambda_3 + 8 - s' - 2r') \\ & - p(\lambda_1 + \lambda_3 + 7 - s' - r', \lambda_1 + \lambda_2 + 7 - s' - 2r') \\ & - p(\lambda_1 + \lambda_2 + 6 - s' - r', \lambda_2 + \lambda_3 + 9 - s' - 2r') \\ & + p(\lambda_2 + \lambda_3 + 8 - s' - r', \lambda_1 + \lambda_2 + 7 - s' - 2r') \\ & + p(\lambda_1 + \lambda_3 + 7 - s' - r', \lambda_2 + \lambda_3 + 9 - s' - 2r') \\ & - p(\lambda_2 + \lambda_3 + 8 - s' - r', \lambda_1 + \lambda_3 + 8 - s' - 2r'), \end{aligned}$$

where $p(\cdot,\cdot)$ is as in (12).

Admittedly, we lack a more conceptual description. The whole branching problem from $Spin_7$ to G_2 deserves a better approach. In fact, a more algebraic study of this kind naturally lands itself in the realm of invariant theory. Such an approach is developed by late Yui Kwan Wong in his 1995 Yale thesis [Won]. A more geometric approach leading to a cleaner formula using Borel-Weil-Bott theorem is taken by McGovern in [McG]. All of these formulas are different than each other.

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