Proceedings of 20^{th} Gökova
Geometry-Topology Conference
pp. 147 – 151

Plenty of Morse functions by perturbing with sums of squares

Antonio Lerario

ABSTRACT. We prove that given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and a submanifold $M \subset \mathbb{R}^n$, then the set of $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that $(f + q_a)|_M$ is Morse, where $q_a(x) = a_1x_1^2 + \cdots + a_nx_n^2$, is residual in $\mathbb{R}^n$. The classical literature covers perturbations by linear functions and quadratic ones but doesn't give an answer to the case of sums of squares: in fact standard transversality arguments do not work and we need a more refined approach.

1. Introduction

The aim of this short note is to answer the following natural question in classical Morse theory; the author was led to consider this problem by K. Kurdyka and A. A. Agrachev, to whom we express our gratitude for stimulating discussions.

"Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and a submanifold $M \subset \mathbb{R}^n$, is a generic perturbation of f by sums of squares Morse on M?"

It turns out that despite its simpleness, this question is in fact more subtle than it seems. Even the classical literaure [1, 2, 3, 4] does not give an answer to it or moves around the obstacle by requiring that M *does not* intersect the coordinate axes - which is in fact what prevents from using the standard transversality techniques; see for example the statement of Corollary 1.25 of [4].

A basic fact of Morse theory is that that given a submanifold $M \subset \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, then the generic perturbation of f with a *linear form* is Morse on M, i.e., the set of vectors $a \in \mathbb{R}^n$ for which $x \mapsto f(x) + \langle a, x\rangle$ is Morse on M is dense in $\mathbb{R}^n$ (see Proposition 17.18 of [1] or Corollary 1.25 of [4]).

It is a well known result that also allowing perturbations of f by *quadratic forms* generically gives a Morse function. In fact typically one can consider the smooth map $F^{\text{ev}} : \text{Sym}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ given by:

$$F^{\text{ev}} : (Q, x) \mapsto \frac{1}{2}Qx + \nabla f(x),$$

Key words and phrases. Morse theory, singularity theory.

The research leading to these results has received funding from the European Community's Seventh Framework Programme ([FP7/2007-2013] [FP7/2007-2011]) under grant agreement no [258204].

(for simplicity we have taken $M = \mathbb{R}^n$). Since F^{ev} is transversal to the zero in $\mathbb{R}^n$, then the *parametric transversality theorem* (Theorem 2.7 in [2] or Theorem 1.21 in [4]) ensures the set of Q for which $x \mapsto \langle x, Qx\rangle + f(x)$ is Morse is dense in $\mathrm{Sym}(n, \mathbb{R})$ (the gradient of the latter function being equal to $F^{\mathrm{ev}}(Q, x)$). In fact one can even reduce to consider perturbation by positive definite forms and the same result holds.

The situation dramatically changes if we allow only perturbations by *sums of squares*. Specifically, given $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ we define the function $q_a : \mathbb{R}^n \to \mathbb{R}$ by:

$$q_a(x) = a_1 x_1^2 + \cdots + a_n x_n^2.$$

If the manifold M *does not* pass through any of the coordinate axes, i.e., it is contained in an open quadrant, we see that the functions $x_1^2, \dots, x_n^2$ can be taken as coordinates on this quadrant and the generic perturbation of f by q_a results into a Morse function, arguing again as above (this is the classical assumption if one wants to perturb by sums of squares).

On the other hand one often deals with situations where the problem is not invariant by translations and the zero (or the axes) plays some special role: that is why the condition that none of the coordinates can vanish on M seems to restrictive.

It turns out that in fact the condition on the relative position of M with respect to the axes is not needed, as we will show in the following theorem.

Theorem 1.1. *Let f be a smooth function on $\mathbb{R}^n$ and $M \subset \mathbb{R}^n$ be a submanifold. Then the set $A(f, M) = \{a \in \mathbb{R}^n \mid (f + q_a)|_M$ is Morse$\}$ is residual in $\mathbb{R}^n$.*

In particular the answer to the above question is affirmative. Notice that if f is defined only on M and M is assumed compact, then we can extend it to the all $\mathbb{R}^n$ and the theorem still applies; moreover, as it will follow from the proof, in the compact case $A(f, M)$ is in fact an open dense set.

It is interesting to understand what makes the parametric transversality argument fail. Following the above ideas we are led to consider the map $F^{\mathrm{ev}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by:

$$F^{\mathrm{ev}} : (a, x) \mapsto \frac{1}{2}\langle a, x\rangle + \nabla f(x)$$

(again for simplicity we take $M = \mathbb{R}^n$). To check the transversality of F^{ev} to the zero in $\mathbb{R}^n$ we compute its differential:

$$dF^{\mathrm{ev}}_{(a,x)}(\dot{a}, \dot{x}) = \frac{1}{2}\langle \dot{a}, x\rangle + \frac{1}{2}\langle a, \dot{x}\rangle + \langle \mathrm{He}_x(f)\dot{x}, \dot{x}\rangle.$$

We see now that *a priori* this differential can have rank smaller than n on the preimage of zero; take for example $f \equiv 0$ and the point $(a, x) = (0, a_2, \dots, a_n, 0, \dots 0)$: then $F^{\mathrm{ev}}(a, x) = 0$ but $\mathrm{rk}(dF^{\mathrm{ev}}_{(a,x)}) < n$.

The paper is structured as follows: we first prove some auxiliary results in Section 2 and Section 3 is devoted to the proof of the main theorem. The techniques we use combine ideas from real algebraic geometry and differential topology. In particular the proof of Theorem 1.1 and its preceding lemmas suggest possible generalizations to o-minimal geometry.

2. Preliminary reductions

We start by recalling the following (Lemma 17.17 from [1]).

Lemma 2.1. *Let f be a smooth function on $\mathbb{R}^n$ and for $a \in \mathbb{R}^n$ define the function f_a by $x \mapsto f(x) + a_1x_1 + \ldots + a_nx_n$. The set:*

$$\{a \in \mathbb{R}^n \mid f_a \text{ is Morse}\}$$

has full measure in $\mathbb{R}^n$.

Proof. Define the function $g(x) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and notice that the Hessian of f is precisely the Jacobian of g and that x is a nondegenerate critical point for f if and only if $g(x) = 0$ and the Jacobian $J_x(g)$ of g at x is nonsingular. Then $g_a(x) = g(x) + a$ and $J(g_a) = J(g)$. We have that x is a critical point for f_a if and only if $g(x) = -a$; moreover it is a nondegenerate critical point if and only if we also have $J_x(g)$ is nonsingular, i.e., a is a regular value of g. The conclusion follows from Sard's lemma. □

We immediately get the following corollary (this is the statement of Theorem 1.1 in the case M is an open subset of $\mathbb{R}^n$ not intersecting the axes).

Corollary 2.2. *If f is a smooth function on an open subset U of $\mathbb{R}^n$ such that for every $u = (u_1, \ldots, u_n) \in U$ we have $u_i \neq 0$ for all $i = 0, \ldots, n$, then*

$$A(f, U) = \{a \in \mathbb{R}^n \mid f + q_a \text{ is Morse on } U\}$$

has full measure in $\mathbb{R}^n$.

Proof. The functions $u_1^2, \ldots, u_n^2$ are coordinates on U by hypothesis; we let $\tilde{f}$ be the function f in these coordinates (it is defined on a certain open subset W of $\mathbb{R}^n$). Then for every $a \in \mathbb{R}^n$ we have that (using the above notation) $\tilde{f}_a$ is Morse on W if and only if $f + q_a$ is Morse on U and the conclusion follows applying the previous lemma. □

To prove the general statement we need the following intermediate step, which removes the condition on the relative position of $U \subset \mathbb{R}^n$ with respect to the axes.

Lemma 2.3. *Let f be a smooth function on an (arbitrary) open subset U of $\mathbb{R}^n$. Then the set $A(f, U)$ is dense in $\mathbb{R}^n$.*

Proof. For every $I = \{i_1, \ldots, i_j\} \subset \{1, \ldots, n\}$ define:

$$H_I = U \cap \{u_i = 0,\, i \in I\} \cap \{u_k \neq 0,\, k \notin I\}.$$

To simplify notations let $I = \{1, \ldots, j\}$. If $a = (a_1, \ldots, a_n)$ and $a'' = (a_{j+1}, \ldots, a_n)$ then $(q_a)|_{H_I} = (q_{a''})|_{H_I}$ where $q_{a''} : \mathbb{R}^{n-j} \to \mathbb{R}$ is defined as above. By Corollary 2.2 the set

$$A''(f, H_I) = \{a'' \in \mathbb{R}^{n-j} \mid f|_{H_I} + q_{a''} \text{ is Morse on } H_I\}$$

is dense in $\mathbb{R}^{n-j}$. Let $a = (a', a'') \in \mathbb{R}^n$ such that $a'' \in A''(f, H_I)$ and suppose $x \in H_I$ is a critical point of $f + q_a$; then x is also a critical point of $(f + q_a)|_{H_I} = f|_{H_I} + q_{a''}$. Since $a'' \in A''(f, H_I)$ then x belongs to a countable set, namely the set $C_{a''}$ of critical points of $f|_{H_I} + q_{a''}$ (each of these critical point must be nondegenerate by the choice of a''); moreover we have that

$$\mathrm{He}_x(f|_{H_I} + q_{a''}) = \mathrm{He}_x(f|_{H_I}) + \mathrm{diag}(a_{j+1}, \dots, a_n)$$

is nondegenerate. Notice that the Hessian of $f + q_a$ at x is a block matrix of the form

$$\mathrm{He}_x(f + q_a) = \left(\begin{array}{c|c} \mathrm{diag}(a_1, \dots, a_j) + B(x) & C(x) \\ \hline C(x)^T & \mathrm{He}_x(f|_{H_I} + q_{a''}) \end{array} \right).$$

Thus for every $a'' = (a_{j+1}, \dots, a_n) \in A''(f, H_I)$ and for every $x \in C_{a''}$ consider the polynomial $p_{a'',x} \in \mathbb{R}[t_1, \dots, t_j]$ defined by:

$$p_{a'',x}(t_1, \dots, t_j) = \det(\mathrm{He}_x(f) + \mathrm{diag}(t_1, \dots, t_j, a_{j+1}, \dots, a_n)).$$

Then the term of maximum degree of $p_{a'',x}$ is

$$t_1 \cdots t_j \det(\mathrm{He}_x(f|_{H_I} + q_{a'}))$$

which is nonzero since $\det(\mathrm{He}_x(f|_{H_I} + q_{a''})) \neq 0$ (x is a *nondegenerate* critical point of $f|_{H_I} + q_{a''}$). It follows that $p_{a'',x}$ is not identically zero; hence its zero locus is a *proper* algebraic set in $\mathbb{R}^j$.
Thus for each $a'' \in A''(f, H_I)$ and each $x \in C_{a''}$ the set $A'(a'', x, I)$ defined by

$$\{a' \in \mathbb{R}^j \,|\, \text{if } x \text{ is a critical point of } f + q_{(a',a'')} \text{ on } H_I \text{ then it is nondegenerate}\}$$

is dense in $\mathbb{R}^j$ (it is the complement of a proper algebraic set). In particular

$$A'(a'', I) = \{a' \in \mathbb{R}^j \,|\, \text{each critical point of } f + q_{(a',a'')} \text{ on } H_I \text{ is nondegenerate}\}$$

also is dense in $\mathbb{R}^j$, since it is a countable intersection of dense sets, i.e.,

$$A'(a'', I) = \bigcap_{x \in C_{a''}} A'(a'', x, I).$$

Thus the set

$$A(f, I) = \{(a', a'') \,|\, a'' \in A''(f, H_I),\ a' \in A'(a'', I)\}$$

(which coincides with the set of $a = (a', a'') \in \mathbb{R}^n$ such that each critical point of $f + q_a$ on H_I is nondegenerate) is dense: for every a'' in the dense set $A''(f, I)$ the set of a' such that $(a', a'') \in A(f, I)$ is dense. Finally

$$A(f, U) = \bigcap_{I \subset \{1, \dots, n\}} A(f, I)$$

is a finite intersection of dense sets, hence dense. $\square$

3. Proof of Theorem 1.1

Proof. Let $u_1, \dots, u_n : \mathbb{R}^n \to \mathbb{R}$ be the coordinates on $\mathbb{R}^n$. Suppose M is of dimension m. For every point $\overline{x} \in M$ there exists a neighborhood W of $\overline{x}$ in M such that $u_{i_1}, \dots, u_{i_m}$ are coordinates for M on

$$W \simeq \mathbb{R}^m,$$

for some $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$; since M is second countable, then it can be covered by a countable (finite if M is compact) number of such open sets. For convenience of notations suppose $\{i_1, \dots, i_m\} = \{1, \dots, m\}$.
Thus $u_1, \dots, u_m$ are coordinates on $W \simeq \mathbb{R}^m$ and $f|_W, u_{m+1}|_W, \dots, u_n|_W$ are functions of $u_1|_W, \dots, u_m|_W$. Fix $a'' = (a_{m+1}, \dots, a_n) \in \mathbb{R}^{n-m}$ and define $g_{a''} : W \to \mathbb{R}$ by

$$g_{a''} = f|_W + a_{m+1}u_{m+1}^2|_W + \cdots + a_n u_m^2|_W = (f + a_{m+1}u_{m+1}^2 + \cdots + a_n u_n^2)|_W$$

Notice that $g_{a''}$ is not $(f + q_a)|_W$ since we are taking only the last $n - m$ of the $a_i's$; we still have the freedom of choice $(a_1, \dots, a_m)$.
By lemma 2.3, since $u_1|_W, \dots, u_m|_W$ are coordinates on W, for every $a'' \in \mathbb{R}^{n-m}$ the set

$$\{a' = (a_1, \dots, a_m) \in \mathbb{R}^m \quad \text{s.t.} \quad g_{a''} + a_1u_1^2|_W + \cdots + a_m u_m^2|_W \text{ is Morse on } W\}$$

is dense in $\mathbb{R}^m$. Notice that $g_{a''} + a_1u_1^2|_W + \cdots + a_m u_m^2|_W = (f + q_{(a',a'')})|_W$; hence for every a'' the set of a' such that $(f + q_{(a',a'')})|_W$ is Morse on W is dense. Thus the set of $a \in \mathbb{R}^n$ such that $(f + q_a)|_W$ is Morse on W is dense (it is dense in a' for each fixed a'' hence it is globally dense). It follows that $A(f, M)$ is a countable intersection of dense sets, hence dense.

Let us now cover M with a countable union of compact sets: $M = \bigcup B_n$; since the map $a \mapsto (f + q_a)|_{B_n}$ is continuous in the Whitney topology and the set of Morse functions is open, then the set of a such that $(f + q_a)|_{B_n}$ is Morse is open (and dense by the previous part).

In particular the set $A(f, M) = \bigcap_n A(f, B_n)$ is a countable intersection of open dense sets, hence residual. $\square$

Acknowledgements: The author wishes to thank A. A. Agrachev, K. Kurdyka and L. Nicolaescu for many interesting discussions.

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.
[2] M. W. Hirsch, *Differential Topology*, Springer-Verlag, 1997.
[3] J. W. Milnor, *Morse Theory*, Annals of Mathematical studies, Princeton University Press, 1969.
[4] L. Nicolaescu, *An invitation to Morse theory*, Springer, 2011.

Institut Camille Jordan, Université Claude Bernard Lyon 1, Lyon, France
E-mail address: `lerario@math.univ-lyon1.fr`