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Isotoping 2-spheres in 4-manifolds

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Abstract. Here we discuss an example of a pair of topologically isotopic but smoothly non-isotopic 2-spheres in a simply connected 4-manifold, which become smoothly isotopic after stabilizing by connected summing with $S^2 \times S^2$, and relate this to a cork twisting operation.

1. The example

In [AKMR], among other things, the authors give an example of topologically isotopic but smoothly non-isotopic spheres in a simply connected 4-manifold, which become smoothly isotopic after connected summing with $S^2 \times S^2$. In this note we show that such an example already follows from [A2].

First we review [A2]: Let $f : \partial W \to \partial W$ be the cork twisting involution of the Mazur cork (W, f). Since f fixes the boundary ∂D of a properly imbedded disk $D \subset W$ up to isotopy (as shown in Figure 1), $f(\partial D)$ bounds a disk in W as well (the isotopy in the collar union D), hence we can extend f across the tubular neighborhood of $N(D)$ of D by the carving process of [A1]. This provides a manifold $Q = W - N(D)$ of Figure 1, homotopy equivalent to $B^3 \times S^1$, and an involution on its boundary $\tau : \partial Q \to \partial Q$.

τ does not extend to Q as a diffeomorphism (otherwise f would extend to a self diffeomorphisim of W). So τ gives an exotic structure to Q relative to its boundary (just as in the cork case). In [A4] such (Q, τ)'s are called *anticorks* because they live inside of corks (W, f), and twisting Q by the involution τ undoes the effect of twisting W by f. Notice that the loop $\gamma = \partial D$ of Figure 1 bounds two different disks in B^4 with the same complement Q (where the identity map between their boundaries can not extend to a diffeomorphism inside), they are described by the two different ribbon moves indicated in the last picture of Figure 1. The two disks are the obvious disks which γ bounds in the third picture of Figure 1, and the same disk after zero and dot exchanges of the figure.

Now let M be the 4-manifold obtained by attaching a 2-handle to B^4 along the ribbon knot γ of Figure 1, with $+1$ framing. Clearly M has two imbedded 2-spheres S_i, $i = 1, 2$ of self intersection $+1$ generating $H_2(M) \cong \mathbb{Z}$, corresponding to the two different 2-disks which γ bounds in B^4. Blowing down either S_1 or S_2 turns M into the *positron* cork $\bar{W}_1$ of Figure 2 ([AM]), and the two different blowing down processes turn the identity map $\partial M \to \partial M$ to the cork involution $f : \partial \bar{W}_1 \to \partial \bar{W}_1$, i.e., the maps in Figure 2 commute (this can be seen by blowing down γ of Figure 1 by using the two different disks). Hence

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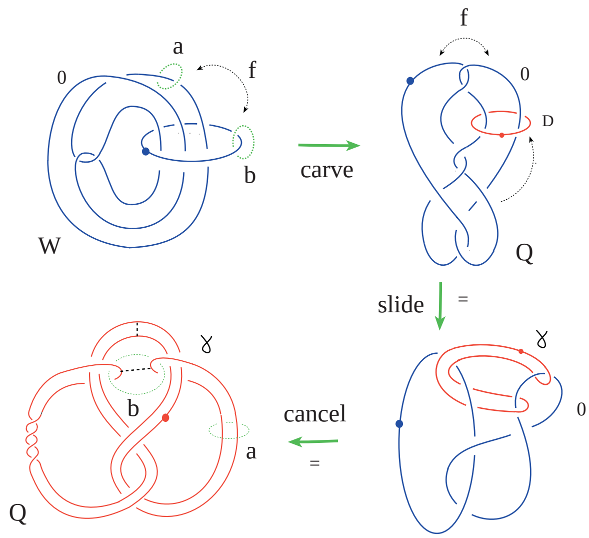

FIGURE 1

S_1 and S_2 are not smoothly isotopic in M by any isotopy keeping ∂M fixed, though they are topologically isotopic (by Freedman's theorem); but they are isotopic In $M\#S^2 \times S^2$ relative to boundary (since surgery corresponds to turning the dotted circle to a 0-framed circle, in the third picture of Figure 1).

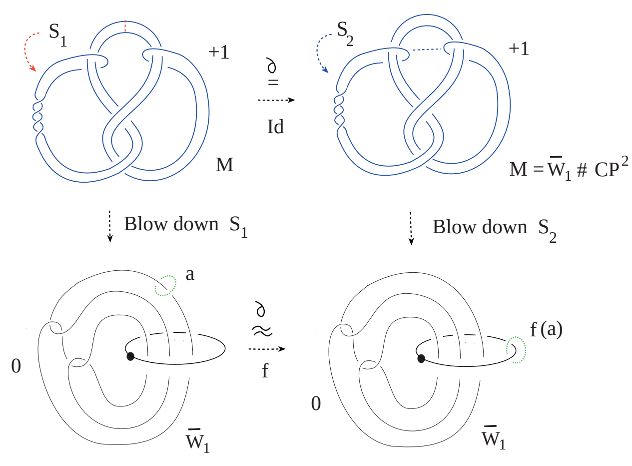

FIGURE 2

Since $M = \bar{W}_1 \# \mathbb{CP}^2$, this example shows that the operation of blowing up $\mathbb{CP}^2$ undoes the cork twisting operation of $(\bar{W}_1, f)$. Also since the Dolgachev surface $E(1)_{2,3}$ differs from its standard copy $\mathbb{CP}^2 \# 9\bar{\mathbb{CP}}^2$ by twisting the positron cork $(\bar{W}_1, f)$ inside (Theorem 1 of [A3]), the manifold M in this example can be made to be closed (without boundary).

Remark 1.1. The reader can check that the two ribbon disks of the ribbon knot in Figure 1 are actually the same ribbon disks (isotoping the last picture of Figure 1, by forcing the two strands going through the circle b stay parallel, results the same picture except the positions of a and b are exchanged) but f induces nontrivial identifications on the boundaries of the ribbon complements.

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