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The space of paths in complex projective space with real boundary conditions

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Abstract. We compute the homology of the space of paths in $\mathbb{C}P^n$ with endpoints in $\mathbb{R}P^n$, $n \geq 1$ and its algebra structure with respect to the Pontryagin-Chas-Sullivan product with $\mathbb{Z}/2$-coefficients. In the orientable case (n odd) we also compute its integral homology. Our method combines Morse theory with geometry and yields an explicit description of cycles representing all homology classes.

1. Introduction

Let $\mathcal{P}_n = \mathcal{P}_{\mathbb{R}P^n}\mathbb{C}P^n$, $n \geq 1$ be the space of continuous paths $\gamma : [0,1] \to \mathbb{C}P^n$ with endpoints $\gamma(0), \gamma(1) \in \mathbb{R}P^n$ and denote

$$\mathbb{H}_\cdot(\mathcal{P}_n) := H_{\cdot+n}(\mathcal{P}_n; \mathbb{Z}/2).$$

More generally, given a $\mathbb{Z}$-graded module $V_\cdot$ and $k \in \mathbb{Z}$ we denote $V[k]_\cdot$ or $V_\cdot[k]$ the $\mathbb{Z}$-graded module $V[k]_j = V_j[k] := V_{j+k}$. Thus $\mathbb{H}_\cdot(\mathcal{P}_n) = H_\cdot(\mathcal{P}_n)[n]$.

We shall refer to $\mathbb{H}_\cdot(\mathcal{P}_n)$ as the *path homology* of the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$ and use for readability the shorthand notation $\mathbb{H}$. This is a unital graded algebra with respect to the *Pontryagin-Chas-Sullivan product*

$$* : \mathbb{H}_i \otimes \mathbb{H}_j \to \mathbb{H}_{i+j}.$$

This product is heuristically defined by intersecting cycles transversally over the evaluation maps at the endpoints, and then concatenating. We refer to §1.2 below for the precise definition. Our grading convention is such that $\mathbb{H}_\cdot$ is supported in degrees $\geq -n$. The class of a point has degree $-n$ and the unit has degree 0, being represented by the cycle $[\mathbb{R}P^n]$ of constant loops.

The purpose of this paper is to compute the algebra $(\mathbb{H}_\cdot, *)$. The result is contained in Theorem 1.1. Along the way, we shall also compute in §3 the integral homology groups $H_\cdot(\mathcal{P}_n; \mathbb{Z})$ for n odd.

Theorem 1.1. *The path homology algebra* $(\mathbb{H}_\cdot(\mathcal{P}_n), *)$, $n \geq 1$ *with* $\mathbb{Z}/2$*-coefficients admits the following presentation:*

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- *if $n \equiv 1$ modulo 4, then*
$$\mathbb{H}_\cdot(\mathcal{P}_n) \simeq \langle H, S, Y \rangle / \left\{ \begin{array}{c} [H,S]=1,\ [H,Y]=0,\ [S,Y]=H^{n-1}Y^2, \\ S^2=0,\ H^{n+1}=0 \end{array} \right\}$$
with $\langle H, S, Y \rangle$ being the free graded unital algebra over $\mathbb{Z}/2$ with generators H, S, Y of degrees
$$|H|=-1, \qquad |S|=1, \qquad |Y|=n.$$
- *if $n \equiv 3$ modulo 4, then*
$$\mathbb{H}_\cdot(\mathcal{P}_n) \simeq \langle H, S, Y \rangle / \left\{ \begin{array}{c} [H,S]=1,\ [H,Y]=0,\ [S,Y]=0, \\ S^2=0,\ H^{n+1}=0 \end{array} \right\}$$
where $\langle H, S, Y \rangle$ has the same meaning as above.
- *if n is even*
$$\mathbb{H}_\cdot(\mathcal{P}_n) \simeq \langle H, T, Y \rangle / \left\{ \begin{array}{c} [H,T]=H,\ [H,Y]=0,\ [T,Y]=Y, \\ T^2=T,\ H^{n+1}=0 \end{array} \right\}$$
with $\langle H, T, Y \rangle$ being the free graded unital algebra over $\mathbb{Z}/2$ with generators H, T, Y of degrees
$$|H|=-1, \qquad |T|=0, \qquad |Y|=n.$$

As far as we know, this is the first computation both of the homology of a space of paths with endpoints on a non-trivial submanifold, and of the product structure that it carries. Our method relies on Morse theory for the energy functional with respect to the Fubini-Study metric and yields a genuine way to visualize all cycles. The key notion is that of a "completing manifold" ([23, IX.7]; Definition 3.1 in §3 below).

Note that the resulting algebra is non-commutative. This is a generic expectation for Pontryagin-Chas-Sullivan algebras of path spaces - in contrast to Chas-Sullivan algebras for loop spaces (see also the section below on Motivations). We find it remarkable that the ring that we compute is as non-commutative as it is, especially given that the fundamental groups are commutative. The most surprising relation is perhaps $SY + YS = H^{n-1}Y^2$ if $n \equiv 1 (\mathrm{mod}\ 4)$, which shows that the induced product on level homology for the energy functional is not commutative either. Here by "level homology" we mean the graded object associated to the filtration of $\mathcal{P}_n$ by sublevel sets of the energy functional (see also [16, §1.2]).

1.1. Motivations and further questions.

The computation that we present was ultimately motivated by the sheer beauty of the structures that emerged. The symmetry of the Fubini-Study metric and the explicit description of geodesics and Jacobi fields on $\mathbb{C}P^n$ yield remarkable properties for the associated energy functional on the space of paths with endpoints on $\mathbb{R}P^n$. Morse theory turns these into a very geometric description of a set of generating cycles.

Beyond that, we view the present paper as lying at the intersection between symplectic topology and string topology. Let M be a closed manifold and N a closed submanifold.

The homology (with local coefficients) of the space $\mathcal{P}_N M$ coincides with the wrapped Floer homology of the conormal bundle $T^*_N M := \{(x,p) \in T^*M \ : \ p|_{T_x N} = 0\}$ [1, 4]. (This is an exact Lagrangian submanifold in T^*M.) Although the Pontryagin-Chas-Sullivan product that we define in this paper has not been studied outside the fundamental case $N = pt$, this isomorphism is expected to intertwine it with the (half-)pair-of-pants product on wrapped Floer homology, in the spirit of [2]. Our computation can be understood as a computation of the wrapped Floer homology ring of $T^*_{\mathbb{R}P^n}\mathbb{C}P^n$.

This correspondence in turn yields many questions. It is known that the wrapped Fukaya category of the cotangent bundle T^*M is generated by a fiber, and hence its Hochschild homology is the homology of the free loop space [3]. One can ask to what extent the Fukaya category is generated by the conormal bundle of a given submanifold. Tobias Ekholm asked the following question: is the Hochschild homology of the algebra $(\mathbb{H}_., *)$ equal to the homology of the free loop space on $\mathbb{C}P^n$? This is a formality question. What is the A_∞-algebra structure on chains on $\mathcal{P}_n$? This should be tractable, since our description of $\mathcal{P}_n$ via Morse theory is essentially complete (and, as Thomas Kragh pointed out, it should even yield the homotopy theory of $\mathcal{P}_n$). What is the Pontryagin-Chas-Sullivan ring $H_.(\mathcal{P}_{\mathbb{R}P^k}\mathbb{C}P^n)$, $k < n$? Can one establish the relative analogue of the multiplicative spectral sequence in [13]? Can one compute the homology or the algebra structure on $H_.(\mathcal{P}_N M)$ from minimal model data on the pair (M, N)? What about the module structure of $H_.(\mathcal{P}_N M)$ over the Chas-Sullivan ring $H_.(\mathcal{L}M)$, where $\mathcal{L}M$ is the free loop space of M?

Another line of motivation stems from "quantum string topology", which is an ongoing project of the second author [24]. From this perspective, the key point is that $\mathbb{R}P^n$ is a monotone Lagrangian inside $\mathbb{C}P^n$, and the Pontryagin-Chas-Sullivan product that we study in this paper can be further deformed using holomorphic discs with boundary on $\mathbb{R}P^n$.

More generally, we believe that it will be important to incorporate path spaces $\mathcal{P}_N M$ into string topology in a systematic way. The only attempt in this direction that we are aware of is [7].

1.2. The Pontryagin-Chas-Sullivan product

Our definition of the product is a variation of the definitions implemented in [11, 12] for the Chas-Sullivan product. We place ourselves in the context of paths of Sobolev class $W^{1,2}$, so that $\mathcal{P}_n$ is a Hilbert manifold. This does not result in any loss of generality since the groups $\mathbb{H}_i$ are invariant under strengthening the regularity scale of the paths under consideration. Let $\mathrm{ev}_t : \mathcal{P}_n \to \mathbb{R}P^n$, $t = 0, 1$ be the evaluation maps at the endpoints given by $\mathrm{ev}_t(\gamma) := \gamma(t)$. The maps ev_t, $t = 0, 1$ are submersions and the fiber product

$$\mathcal{C}_n := \mathcal{P}_n \ {}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_0} \mathcal{P}_n := \{(\gamma, \delta) \in \mathcal{P}_n \times \mathcal{P}_n \ : \ \mathrm{ev}_1(\gamma) = \mathrm{ev}_0(\delta)\}$$

is a codimension n Hilbert submanifold of $\mathcal{P}_n \times \mathcal{P}_n$. We denote the inclusion

$$s : \mathcal{C}_n \hookrightarrow \mathcal{P}_n \times \mathcal{P}_n.$$

We also consider the *concatenation map (at time $t = 1/2$)*

$$c : \mathcal{C}_n \hookrightarrow \mathcal{P}_n$$

defined by $c(\gamma, \delta)(t) := \gamma(2t)$ for $0 \le t \le 1/2$ and $c(\gamma, \delta)(t) := \delta(2t-1)$ for $1/2 \le t \le 1$. Note that c is a codimension n embedding and its image is the set $\{\gamma \in \mathcal{P}_n \, : \, \gamma(\frac{1}{2}) \in \mathbb{R}P^n\}$. We thus have a diagram

$$\mathcal{P}_n \xleftarrow{c} \mathcal{C}_n \xrightarrow{s} \mathcal{P}_n \times \mathcal{P}_n$$

Restricting to $\mathbb{Z}/2$-coefficients, we obtain the following diagram in homology

$$\mathbb{H}_{i+j} \xleftarrow{c_*} H_{i+j+n}(\mathcal{C}_n) \xleftarrow{s_!} H_{i+j+2n}(\mathcal{P}_n \times \mathcal{P}_n) \xleftarrow{EZ} \mathbb{H}_i \otimes \mathbb{H}_j.$$

Here EZ denotes the Eilenberg-Zilber map, which can be explicitly represented at chain level by the Eilenberg-MacLane (p, q) shuffle ([14, p.64], compare also [21, pp.133-134], [17, p. 268]). The map c_* is the map induced in homology by concatenation. As for the *shriek map* $s_!$, this is defined as the composition

$$H_\cdot(\mathcal{P}_n \times \mathcal{P}_n) \to H_\cdot(\mathcal{P}_n \times \mathcal{P}_n, \mathcal{P}_n \times \mathcal{P}_n \setminus \mathcal{C}_n) \to H_\cdot(\nu(\mathcal{C}_n), \nu(\mathcal{C}_n) \setminus \mathcal{C}_n) \xrightarrow{\tau^{-1}} H_{\cdot-n}(\mathcal{C}_n).$$

Here the first map is induced by inclusion, the second map is given by excision after having identified the normal bundle $\nu(\mathcal{C}_n)$ to $\mathcal{C}_n$ inside $\mathcal{P}_n \times \mathcal{P}_n$ to a tubular neighborhood of $\mathcal{C}_n$, and τ^{-1} is the (inverse of) the Thom isomorphism.

The Pontryagin-Chas-Sullivan product is defined to be

$$* : \mathbb{H}_\cdot \otimes \mathbb{H}_\cdot \to \mathbb{H}_\cdot, \qquad \alpha * \beta := c_* s_! EZ(\alpha \otimes \beta).$$

Remark 1.1 (On the Chas-Sullivan and Pontryagin products)**.** This definition holds more generally for the space $\mathcal{P}_N M$ of paths inside a manifold M with endpoints on a submanifold N. As such, it generalizes the definition of the Chas-Sullivan product: by "folding up" at time $1/2$, free loops inside a manifold P can be viewed as paths inside $P \times P$ with endpoints on the diagonal $\Delta \subset P \times P$, and the Chas-Sullivan product can be expressed in terms of the Pontryagin-Chas-Sullivan product for $\mathcal{P}_\Delta(P \times P)$.

This definition also generalizes that of the Pontryagin product on the based loop space $\Omega M = \mathcal{P}_{pt} M$. This is our motivation for the choice of terminology "Pontryagin-Chas-Sullivan product".

Remark 1.2 (On orientability and integral coefficients)**.** The key ingredient in the definition of the product is the shriek map $s_!$, which in turn relies on the Thom isomorphism

$$H_\cdot(\nu(\mathcal{C}_n), \nu(\mathcal{C}_n) \setminus \mathcal{C}_n) \simeq H_{\cdot-n}(\mathcal{C}_n).$$

The isomorphism holds as such with $\mathbb{Z}/2$-coefficients, and it holds with $\mathbb{Z}$-coefficients if the normal bundle $\nu(\mathcal{C}_n)$ is orientable. In case the normal bundle is not orientable, the isomorphism holds under the form

$$H_\cdot(\nu(\mathcal{C}_n), \nu(\mathcal{C}_n) \setminus \mathcal{C}_n) \simeq H_{\cdot-n}(\mathcal{C}_n; o_{\nu(\mathcal{C}_n)}),$$

where $o_{\nu(\mathcal{C}_n)}$ is the local system of orientations for $\nu(\mathcal{C}_n)$. In our situation we have $\mathcal{C}_n = (\mathrm{ev}_1, \mathrm{ev}_0)^{-1}(\Delta)$ with $(\mathrm{ev}_1, \mathrm{ev}_0) : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{R}P^n \times \mathbb{R}P^n$, and thus

$$\nu(\mathcal{C}_n) = (\mathrm{ev}_1, \mathrm{ev}_0)^* \nu_{\mathbb{R}P^n \times \mathbb{R}P^n} \Delta.$$

Using that $\nu_{\mathbb{R}P^n \times \mathbb{R}P^n} \Delta \simeq T\mathbb{R}P^n$ we obtain that $\nu(\mathcal{C}_n)$ is orientable if and only if n is odd. Moreover, a straightforward computation shows that the Pontryagin-Chas-Sullivan product is defined with integral coefficients on $H_\cdot(\mathcal{P}_n; o_{\mathrm{ev}_0^* T\mathbb{R}P^n})$. For the more general case of the space $\mathcal{P}_N M$, the Pontryagin-Chas-Sullivan product is defined with integral coefficients on $H_\cdot(\mathcal{P}_N M; o_{\mathrm{ev}_0^* TN})$.

1.3. Convention for concatenation.

In the above discussion we have used the concatenation map $c : \mathcal{C}_n \to \mathcal{P}_n$ at time $t = 1/2$. This is of course not associative. However, within the setup of paths of Sobolev class $W^{1,2}$ there is a very elegant way to obtain associativity by using what we will call the *energy minimizing concatenation map*

$$c_{\min} : \mathcal{C}_n \to \mathcal{P}_n.$$

This map appeared for the first time in [16, Lemma 2.4, see also §10.6] and is closely related to the classical *energy functional*

$$E : \mathcal{P}_n \to \mathbb{R}_+, \qquad E(\gamma) := \int_0^1 |\gamma'|^2,$$

and even more to what we call the *(L^2-) norm* functional

$$F : \mathcal{P}_n \to \mathbb{R}_+, \qquad F(\gamma) := \left(\int_0^1 |\gamma'|^2 \right)^{1/2} = \|\gamma'\|_{L^2}.$$

The map $c_{\min}$ is defined by

$$c_{\min}(\gamma, \delta)(t) := \left\{ \begin{array}{c} \gamma(\frac{t}{s}) \text{ if } 0 \le t \le s \\ \delta(\frac{t-s}{1-s}) \text{ if } s \le t \le 1 \end{array} \right\}$$

where

$$s = s_{\min} := \frac{F(\gamma)}{F(\gamma) + F(\delta)}.$$

The key point is that the concatenation product $c_{\min}$ produces the unique piecewise linear concatenation of minimum energy. This implies in turn associativity

$$c_{\min}(c_{\min}(\alpha, \beta), \gamma) = c_{\min}(\alpha, c_{\min}(\beta, \gamma)).$$

This can of course also be checked directly, and we do encourage the reader to perform this surprising calculation. The map $c_{\min}$ is homotopic to the concatenation c and the Pontryagin-Chas-Sullivan product can therefore alternatively be defined using $c_{\min}$.

The map $c_{\min}$ and the norm F satisfy the following remarkable and useful identity:

$$F(c_{\min}(\gamma, \delta)) = F(\gamma) + F(\delta).$$

This is the key to Lemma 4.4 in §4.1, which relates the Pontryagin-Chas-Sullivan product and min-max critical values of the functional F. We would like to emphasize that, from the joint point of view of variational calculus and of product structures on $H.(\mathcal{P}_n)$, the norm F plays a more fundamental role than the energy E, although they have the same Morse theory (same critical points, index, and nullity).

Convention. We shall deal in the sequel with concatenations of arbitrary many paths, and we shall always perform the concatenation using the map $c_{\min}$.

1.4. Structure of the paper.

The paper contains three sections whose names are self-explanatory: Geometry, Homology, and Product. Our main tool is Morse theory for the norm functional F for the Fubini-Study metric on $\mathbb{C}P^n$, and the key point is the existence of completing manifolds in the sense of Definition 3.1.

2. The geometry of $\mathcal{P}_n$

In all subsequent sections we work with paths of Sobolev class $W^{1,2}$ on the space $\mathbb{C}P^n$ equipped with the Fubini-Study metric. A traditional approach to path spaces ([9], [22]) that works well with the finite dimensional approximation of Morse is the Morse theory for the *energy* functional

$$E : \mathcal{P}_n \to \mathbb{R}, \qquad E(\gamma) := \int_0^1 |\gamma'(t)|^2 dt.$$

Following [16] we will use instead the (L^2-) *norm*

$$F : \mathcal{P}_n \to \mathbb{R}, \qquad F(\gamma) := \sqrt{E(\gamma)} = \left(\int_0^1 |\gamma'(t)|^2 dt\right)^{\frac{1}{2}} = \|\gamma'\|_{L^2}.$$

Note that

$$F(\gamma) \geq length(\gamma),$$

with equality if and only if γ is parametrized proportional to arclength; it is a good approximation to the truth to think of F as the length. The norm F is not differentiable on the constant paths; we consider them critical points of F. The functions E and F have (by a simple computation) the same critical points, with the same index and nullity and thus produce the same Morse theory. However the norm F behaves better with respect to concatenation of paths, as explained in §1.3. The path homology algebra of the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$ is of course independent of the metric; it is standard procedure to use Morse theory with a special function having simple critical points to compute topological data.

The critical points of F are the geodesics that are perpendicular to $\mathbb{R}P^n$ at both ends. Geodesics of the Fubini-Study metric have the following simple description [6, Proposition 3.32]: given a point $z \in \mathbb{C}P^n$ and a unit tangent vector $v \in T_z\mathbb{C}P^n$, the geodesic $\gamma_{z,v}(s) := \exp_z(sv)$, $s \in \mathbb{R}$ is periodic of period π and is a great circle parametrized by arc-length on the unique complex line $\ell_{z,v}$ ($\simeq \mathbb{C}P^1$) passing through z and tangent to v.

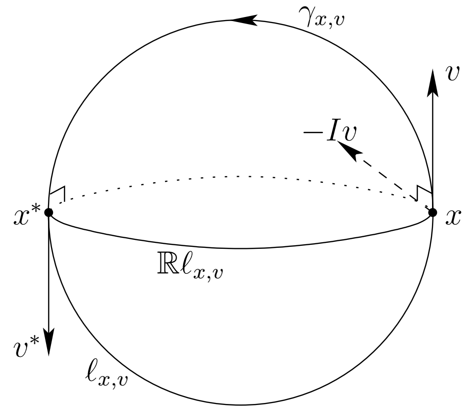


FIGURE 1. Geodesics in $\mathbb{C}P^n$.

Note that the Fubini-Study metric on $\mathbb{C}P^n$ is such that complex lines are round spheres of curvature 4 and circumference π. In particular, if $x \in \mathbb{R}P^n$ and $v \in SN\mathbb{R}P^n$ (the unit normal bundle of $\mathbb{R}P^n$ in $\mathbb{C}P^n$), so that the geodesic $\gamma_{x,v}$ starts on $\mathbb{R}P^n$ in an orthogonal direction, it will meet (orthogonally) again $\mathbb{R}P^n$ at times $s = k\pi/2$, $k \geq 1$. If k is even then $\gamma_{x,v}(k\pi/2) = x$ and $\gamma'_{x,v}(k\pi/2) = v$, whereas if k is odd then $\gamma_{x,v}(k\pi/2) = x^*$, the antipode of x in $\ell_{x,v}$, and $\gamma'_{x,v}(k\pi/2) = v^*$, where v^* is the image of v under the antipodal map on $\ell_{x,v}$. The point x^* can be alternatively described as the cutpoint of x inside $\mathbb{R}P^n$ in the direction $-Iv \in T_x\mathbb{R}P^n$, where $I : T_x\mathbb{C}P^n \to T_x\mathbb{C}P^n$ is the complex structure. Yet another description of x^* is the following: the complex line $\ell_{x,v}$ intersects $\mathbb{R}P^n$ along its equator $\mathbb{R}\ell_{x,v}$, which is the unique real projective line in $\mathbb{R}P^n$ passing through x in the direction Iv. The geodesic $\gamma_{x,-Iv}$ is a parametrization of $\mathbb{R}\ell_{x,v}$ by arc-length and the cutpoint is $\gamma_{x,-Iv}(\pi/2) = x^*$. In fact, the cut-locus of x in $\mathbb{R}P^n$ is a real hyperplane whose intersection with $\mathbb{R}\ell_{x,v}$ is the point x^*, see Figure 1.

It follows from this discussion that the critical set of F is a disjoint union of manifolds K_k, $k \geq 0$ that consist respectively of geodesics of length $k\pi/2$ starting (and ending) on $\mathbb{R}P^n$ perpendicularly. If $k = 0$ then $K_0 = \mathbb{R}P^n$, the space of constant paths. If $k \geq 1$ there is a natural identification of K_k with $SN\mathbb{R}P^n$ given by

$$SN\mathbb{R}P^n \xrightarrow{\simeq} K_k, \qquad (x,v) \mapsto \gamma_{x,v}|_{[0,k\pi/2]}.$$

Of course, the unit normal bundle $SN\mathbb{R}P^n$ is isomorphic to the unit tangent bundle of $\mathbb{R}P^n$, denoted $ST\mathbb{R}P^n$, via

$$SN\mathbb{R}P^n \xrightarrow{\simeq} ST\mathbb{R}P^n, \qquad (x,v) \mapsto (x, -Iv).$$

Recall that, given a function $f : X \to \mathbb{R}$ of class C^2 on a Hilbert manifold X and a submanifold K consisting of critical points of f, we say that K is a *Morse-Bott critical manifold of index* $\iota(K)$ *and nullity* $\eta(K)$ if at each point $p \in K$ the second derivative, or Hessian $H_f(p)$ is nondegenerate on the normal bundle to K in X, and if $H_f(p)$ has index $\iota(K)$ and nullity $\eta(K)$. (The index is the dimension of a maximal subspace of T_pX, $p \in K$ on which $H_f(p)$ is negative definite, and the nullity is the dimension of the null space of $H_f(p)$.) Note that the first condition in the definition of a Morse-Bott critical manifold is equivalent to requiring the tangent space T_pK to coincide with the null-space of $H_f(p)$.

Lemma 2.1. *The manifolds K_k, $k \geq 0$ are Morse-Bott. Their index and nullity are respectively given by*

$$\iota(K_0) = 0, \qquad \eta(K_0) = n$$

and, if $k \geq 1$,

$$\iota(K_k) = 1 + (k-1)n, \qquad \eta(K_k) = 2n - 1.$$

The proof is based on a space of "half-circles" in $\mathbb{C}P^n$ that captures the Morse theory of the norm functional on the space $\mathcal{P}_n$.

Circles and Half-circles in $\mathbb{C}P^n$

We begin with several descriptions of the space of "vertical circles" in $\mathbb{C}P^n$. Given $x \in \mathbb{R}P^n$ and $v \in SN\mathbb{R}P^n$, non-constant and unparametrized circles on the complex line $\ell_{x,v}$ form a well-defined conformally invariant class. (The authors could not resist pointing out that circles on $\mathbb{C}P^1$ (including constant ones) are precisely the images of great circles on the round 3-sphere via the Hopf map. Similarly, circles on $\mathbb{C}P^n$, defined as circles on the complex lines $\ell_{p,v}$, are precisely the images of great circles on the round $2n+1$-sphere via the Hopf map.) As such, if $x' \in \mathbb{R}\ell_{x,v}$, $x' \neq x$, there is a unique circle through x, tangent to v and passing through x'. Note this circle is orthogonal to $\mathbb{R}\ell_{x,v}$ at x and at x' (thus "vertical"), and can also be found as follows: if $x', x \in \mathbb{R}P^n$ with $x' \neq x$, then there is a unique complex line containing them, and a unique circle containing x and x' and meeting $\mathbb{R}P^n$ orthogonally, the unique *vertical* circle containing x and x'.

Yet another description of the vertical circle containing x and x': embed $\ell_{x,v}$ isometrically inside $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ as the sphere of radius $1/2$ with equator $\mathbb{R}\ell_{x,v} = \ell_{x,v} \cap (\mathbb{R}^2 \times \{0\})$ and $v \in \{0\} \times \mathbb{R}^+$. Then x and x' lie on the equator, and the vertical circle is the intersection of the sphere $\ell_{x,v}$ with the unique vertical plane containing x and x', see Figure 2.

For fixed x and v, $\mathbb{R}\ell_{x,v}$ is naturally identified with $S^1_\pi := \mathbb{R}/\pi\mathbb{Z}$ via

$$x' = \exp_x(-\theta I v),$$

$\theta \in S^1_\pi$. (In particular, the antipode x^* of x on $\ell_{x,v}$ corresponds to $\theta = \frac{\pi}{2}$. The sign of θ is chosen so that the equator $\{x'\}$ is parametrized counterclockwise from above.) The principal S^1_π-bundle over $SN\mathbb{R}P^n$ with fiber $\mathbb{R}\ell_{x,v}$ is therefore identified with the product

$$\{(x, v, \theta)\} = SN\mathbb{R}P^n \times S^1_\pi.$$

The parameters x, v, θ also determine a unique *vertical half-circle*, denoted $C_{x,v,\theta}$; the half circle $C_{x,v,\theta}$ is defined to be the intersection of the *upper* hemisphere in $\ell_{x,v}$ (as just embedded in $\mathbb{R}^3$) with the unique vertical plane containing x and x'. We parametrize the half-circle $C_{x,v,\theta}$ on the interval $[0,1]$ with constant speed so that it begins at x and ends

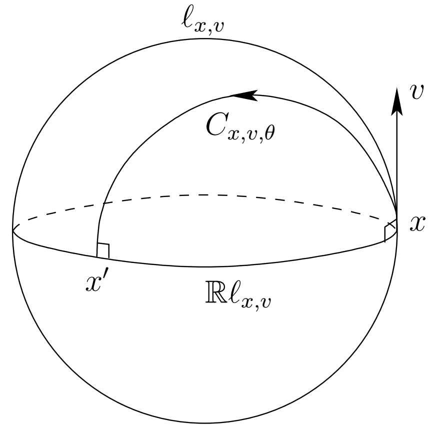


FIGURE 2. Vertical half-circle in $\mathbb{C}P^n$.

at x'. If $x' = x$ we define $C_{x,v,\theta}$ to be the constant circle at x and we think of it as having the tangent vector v attached to it. Thus we have a map

$$C : SN\mathbb{R}P^n \times S^1_\pi \to \mathcal{P}_n, \qquad (x, v, \theta) \mapsto C_{x,v,\theta}.$$

We define

$$Y_1 := SN\mathbb{R}P^n \times S^1_\pi = \{(x, v, \theta)\},$$

which we think of as being the space of vertical half-circles as above. We have smooth evaluation maps ev_0, $\mathrm{ev}_1 : Y_1 \to \mathbb{R}P^n$ defined by

$$\mathrm{ev}_0(p, v, \theta) := C_{x,v,\theta}(0) = x, \qquad \mathrm{ev}_1(x, v, \theta) := C_{x,v,\theta}(1) = x'.$$

Both evaluation maps are submersions. We define

$$Y_k := Y_1 \,{}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_0} Y_1 \,{}_{\mathrm{ev}_1}\!\times \cdots \times_{\mathrm{ev}_0} Y_1,$$

where the number of factors in the fiber product is equal to $k \geq 1$. We think of this as being the space of paths $[0, k] \to \mathbb{C}P^n$ whose restriction to each interval $[j, j + 1]$, $j \in \{0, \ldots, k - 1\}$ is a vertical half-circle C_{x_j,v_j,θ_j}. These half-circles have matching endpoints and, if any of them is constant, it also has a unit normal vector $v_j \in SN\mathbb{R}P^n$ attached to it. We denote x_k the endpoint of $C_{x_{k-1},v_{k-1},\theta_{k-1}}$. Note that two distinct points $x, y \in \mathbb{R}P^n$ determine a unique complex line $\ell_{x,y} \subset \mathbb{C}P^n$, a unique vertical circle on $\ell_{x,y}$ passing through x and y, and thus *two* (unparametrized) vertical half-circles with endpoints x and y. As a consequence, the manifold Y_k is naturally parametrized near an element with x_j distinct from x_{j+1} for all $j \in \{0, \ldots k - 1\}$ by the sequence $(x_0, x_1, \ldots, x_k) \in (\mathbb{R}P^n)^{k+1}$. In particular the open subset of paths in Y_k with distinct

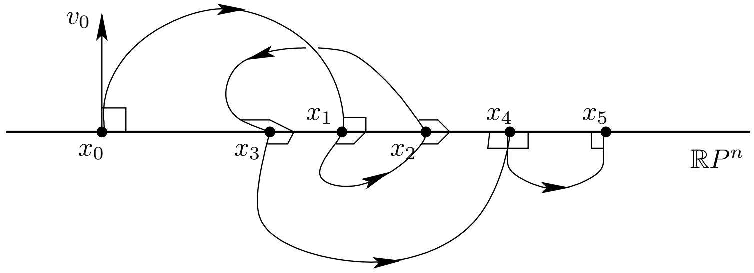


FIGURE 3. An element of Y_5.

adjacent x_j *embeds* in $\mathcal{P}_n$ and is locally diffeomorphic to an open set in $(\mathbb{R}P^n)^{k+1}$. This is consistent with the fact that

$$\dim Y_k = (k+1)n.$$

The spaces Y_k, $k \geq 1$ have the following important features:

(i) there is a natural map
$$\varphi_k : Y_k \to \mathcal{P}_n \tag{1}$$
defined by concatenating the vertical half-circles that constitute an element of Y_k. More precisely, we have
$$\varphi_k\big((x_0, v_0, \theta_0), \ldots, (x_{k-1}, v_{k-1}, \theta_{k-1})\big) := C_{x_0,v_0,\theta_0} \cdot C_{x_1,v_1,\theta_1} \cdot \ldots \cdot C_{x_{k-1},v_{k-1},\theta_{k-1}}.$$

(ii) the critical set K_k naturally embeds into Y_k with codimension $1+(k-1)n = \iota(K_k)$ via the map
$$\gamma_{x,v}|_{[0,k\pi/2]} \mapsto \big((x, v, \frac{\pi}{2}), (x', v', \frac{\pi}{2}), (x, v, \frac{\pi}{2}), \ldots\big), \tag{2}$$
where (x', v') is the image of (x, v) under the derivative of the antipodal map on $\ell_{x,v}$. We denote
$$L_k \subset Y_k$$
the image of this embedding. Note that φ_k is itself an embedding near L_k.

(iii) the image of Y_k in $\mathcal{P}_n$ contains piecewise geodesics that are not smooth. These paths are not critical points of F, but the value of F at these points is the maximum value $k\pi/2$. However, after a small perturbation of the map φ_k in the direction $-\nabla F$, the image of $Y_k \setminus L_k$ will be contained in the set $\{F < k\pi/2\}$.

We will see in §3 that, due to the above properties, the spaces of vertical half-circles Y_k faithfully reflect the Morse theory of F and actually carry the topology of $\mathcal{P}_n$. The relevant notion is that of a *completing manifold* (see Definition 3.1 in the next section).

Proof of Lemma 2.1. The statement concerning K_0 is a general fact, for which we refer to [20, Proposition 2.4.6]. We now focus on the case $k \geq 1$.

That $\eta(K_k) = 2n-1$ follows from an explicit computation of which we only give a sketch and omit the details. The null-space of E (or F) at a critical point $\gamma \in \mathcal{P}$ is spanned by Jacobi fields along γ whose covariant derivative at the endpoints is orthogonal to $\mathbb{R}P^n$. (The explicit formula for the second derivative of E can be found in [19, p. 663] and it implies this condition in view of the fact that $\mathbb{R}P^n$ is totally geodesic inside $\mathbb{C}P^n$.) Since one has explicit formulas for the Jacobi fields along geodesics for the Fubini-Study metric on $\mathbb{C}P^n$ (see for example [6, Proposition 3.34] or [15, pp. 125-126]), the result follows readily. See also Remark 2.1 below.

Since
$$\eta(K_k) = \dim K_k = 2n-1$$
(and since clearly the index is constant on the critical set K_k), we infer that K_k is a nondegenerate Morse-Bott manifold. The manifold Y_k is embedded near K_k. Since $F = k\pi/2$ on K_k and $F \leq k\pi/2$ on Y_k , it follows that the second derivative of F is ≤ 0 on the normal bundle to K_k in Y_k, and at each point γ of K_k there is a subspace of $T_\gamma \mathcal{P}_n$ of dimension $\dim Y_k = (k+1)n$ on which the second derivative of F is ≤ 0. Thus $\iota(K_k) + \eta(K_k) \geq (k+1)n$. But since we know the nullity, it must be that
$$\iota(K_k) \geq \operatorname{codim}(K_k) = 1 + (k-1)n.$$

We will now prove the reverse inequality
$$\iota(K_k) \leq 1 + (k-1)n, \tag{3}$$
from which the Lemma follows. Our proof of (3) goes by comparing the energy functional on $\mathcal{P}_n$ with the energy functional on the space of free loops in $\mathbb{C}P^n$ and making use of the following well-known facts about the latter. Denote $\mathcal{L}\mathbb{C}P^n := \{\gamma : S^1_\pi \to \mathbb{C}P^n\}$ the space of free loops of Sobolev class $W^{1,2}$ on $\mathbb{C}P^n$. The critical set of the energy functional on $\mathcal{L}\mathbb{C}P^n$ for the standard metric is a disjoint union of Morse-Bott nondegenerate manifolds L_m, $m \geq 0$ which consist respectively of the geodesics of length $m\pi$ (we refer to these as being geodesics of *multiplicity* m). The index and nullity of these critical manifolds are respectively equal to $\iota(L_0) = 0$, $\eta(L_0) = 2n$ and, for $m \geq 1$, [25, §2.2]
$$\iota(L_m) = 1 + (m-1)2n, \qquad \eta(L_m) = 4n-1. \tag{4}$$

Proof of (3) *for k odd.* Since inequality (3) depends only on the behavior of E near K_k, let us fix a small enough neighborhood of K_k in $\mathcal{P}_k$, denoted $\mathcal{N}_k$. There is an embedding
$$\Phi_k : \mathcal{N}_k \hookrightarrow \mathcal{L}\mathbb{C}P^n$$
that associates to a path γ with endpoints x_0 and x_k the loop $\gamma \cdot_{\min} C_{x_k,x_0}$, where $\cdot_{\min}$ denotes the minimal energy concatenation, and C_{x_k,x_0} is the unique vertical half-circle joining the two distinct points x_k and x_0 such that $\gamma \cdot_{\min} C_{x_k,x_0}$ is close to a closed geodesic at x_0 in $L_{\frac{k+1}{2}}$. Note that $\Phi_k(K_k) = L_{\frac{k+1}{2}}$. If $V \subset T_\gamma \mathcal{P}_n$ is a subspace on which the second derivative of F is negative definite, $d\Phi_k V$ is a subspace of $T_{\Phi_k(\gamma)}\mathcal{L}\mathbb{C}P^n$ of the same dimension (because Φ_k is an embedding); since
$$F(\Phi_k(\gamma)) = F(\gamma) + \frac{\pi}{2}$$

and

$$F(\Phi_k(\tau)) = F(\tau) + F(C_{x_k,x_0}) \leq F(\tau) + \frac{\pi}{2}$$

for all $\tau \in \mathcal{N}_k$, the second derivative of F on $d\Phi_k V$ is also negative definite, and therefore the index of a geodesic $\gamma \in K_k$ in $\mathcal{P}_n$ is bounded from above by the index of the geodesic $\Phi_k(\gamma) \in L_{\frac{k+1}{2}}$ in $\mathcal{L}\mathbb{C}P^n$, which is equal to $1+(\frac{k+1}{2}-1)2n = 1+(k-1)n$. This proves (3).

Proof of (3) *for k even.* Given a point $* \in \mathbb{R}P^n$ we denote $\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n$ the space of paths $\gamma : [0,1] \to \mathbb{C}P^n$ of Sobolev class $W^{1,2}$ such that $\gamma(0) = *$ and $\gamma(1) \in \mathbb{R}P^n$. Let $\Omega\mathbb{C}P^n$ be the space of loops of Sobolev class $W^{1,2}$ in $\mathbb{C}P^n$ based at $*$. The obvious inclusions

$$\Omega\mathbb{C}P^n \subset \mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n \subset \mathcal{P}_n$$

are codimension n embeddings. Note that any element of K_k belongs to $\mathcal{L}\mathbb{C}P^n$, and also to $\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n$ and $\Omega\mathbb{C}P^n$ for a suitable choice of basepoint, and is a critical point of the restriction of the norm (and the energy function) to each of these spaces. Given $\gamma \in K_k$ we denote $\iota_{\mathcal{P}}(\gamma)$ the index of γ as a critical point of the norm restricted to the submanifold $\mathcal{P}$ of the space of paths on $\mathbb{C}P^n$. We prove the following relations:

$$\iota_{\mathcal{P}_n}(\gamma) \leq \iota_{\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n}(\gamma), \tag{5}$$

$$\iota_{\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n}(\gamma) \leq \iota_{\Omega\mathbb{C}P^n}(\gamma) + n, \tag{6}$$

$$\iota_{\Omega\mathbb{C}P^n}(\gamma) = \iota_{\mathcal{L}\mathbb{C}P^n}(\gamma). \tag{7}$$

Since $\gamma \in K_k \subset \mathcal{L}\mathbb{C}P^n$ has multiplicity $m = k/2$, we obtain using (4) that $\iota_{\mathcal{L}\mathbb{C}P^n}(\gamma) = 1 + (\frac{k}{2} - 1)2n = 1 + (k-2)n$. Thus $\iota_{\mathcal{P}_n}(\gamma) \leq 1 + (k-2)n + n = 1 + (k-1)n$, which proves (3).

Let us prove (5). The group $\mathrm{Isom}(\mathbb{C}P^n, \mathbb{R}P^n)$ of isometries of $\mathbb{C}P^n$ that preserve $\mathbb{R}P^n$ acts transitively on $\mathbb{R}P^n$. Let us choose an open neighborhood U of $*$ in $\mathbb{R}P^n$ and a smooth map

$$\psi : U \to \mathrm{Isom}(\mathbb{C}P^n, \mathbb{R}P^n)$$

such that

$$\psi(x)(x) = *.$$

Recall the evaluation map $\mathrm{ev}_0 : \mathcal{P}_n \to \mathbb{R}P^n$ and let $\mathcal{N} := \mathrm{ev}_0^{-1}(U) \subset \mathcal{P}_n$, so that $\mathcal{N}$ is an open neighborhood of $\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n$. We have diffeomorphisms which are inverse to each other

$$\mathcal{N} \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n \times U$$

given by

$$P(\alpha) := (\psi(\mathrm{ev}_0(\alpha)) \circ \alpha, \mathrm{ev}_0(\alpha)), \qquad Q(\beta, x) := \psi(x)^{-1} \circ \beta.$$

Since we use elements $\psi \in \mathrm{Isom}(\mathbb{C}P^n, \mathbb{R}P^n)$ the norm is preserved in the sense that, if $\alpha \in \mathcal{N}$, then

$$F(\alpha) = F \circ pr_1 \circ P(\alpha)$$

where pr_1 is the projection on the first factor of $\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n \times U$. Now let $V \subset T_\gamma\mathcal{N}$ be a subspace of dimension ι on which the second derivative of the norm is negative definite. Then $d(pr_1 \circ P)(V)$ is a subspace of $\mathrm{T}_\gamma\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n$ of dimension ι on which the second derivative is negative definite. This implies that $\iota_\mathcal{N}(\gamma) \leq \iota_{\mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n}(\gamma)$, which is equivalent to (5).

Inequality (6) is implied by the fact that the embedding $\Omega\mathbb{C}P^n \subset \mathcal{P}_{*,\mathbb{R}P^n}\mathbb{C}P^n$ has codimension n.

The identity (7) is proved by an argument similar to the one given for (5): using the fact that the group $\mathrm{Isom}(\mathbb{C}P^n)$ of isometries of $\mathbb{C}P^n$ acts transitively, we construct a norm preserving diffeomorphism $\mathcal{N} \simeq \Omega\mathbb{C}P^n \times U$ between an open neighborhood $\mathcal{N} := \mathrm{ev}_0^{-1}(U)$ of $\Omega\mathbb{C}P^n$ in $\mathcal{L}\mathbb{C}P^n$ and the product of $\Omega\mathbb{C}P^n$ with an open neighborhood U of $*$ in $\mathbb{C}P^n$. The relation $F|_\mathcal{N} \equiv F|_{\Omega\mathbb{C}P^n} \circ pr_1$ implies (7). □

Remark 2.1 (Morse index theorem for the endmanifold case)**.** The statement of the above Lemma 2.1 can alternatively be proved using the Morse index theorem for the endmanifold case in [19]. The key notions are those of $\mathbb{R}P^n$-*focal point* and $\mathbb{R}P^n$-*Jacobi field* along a geodesic γ starting at $x \in \mathbb{R}P^n$ in an orthogonal direction. These generalize the standard notions of conjugate point and Jacobi field. Since the Jacobi fields in $\mathbb{C}P^n$ for the Fubini-Study metric are explicitly known [15, pp. 125-126], the $\mathbb{R}P^n$-focal points can also be computed explicitly. The key argument in this computation is the following Lemma, which we leave to the interested reader. Given a set $\mathcal{S}$ we denote by $\mathrm{Vect}(\mathcal{S})$ the vector space generated by $\mathcal{S}$.

Lemma 2.2. *Let $x \in \mathbb{R}P^n$ and γ be a geodesic in $\mathbb{C}P^n$ starting at x with unit speed in a direction orthogonal to $\mathbb{R}P^n$. Denote $X_1 := \dot\gamma(0)$, $X_2 := IX_1$, let $(X_3, X_5, \ldots X_{2n-1})$ be an orthonormal basis of $\langle X_1\rangle^\perp$ in $T_x\mathbb{R}P^n$ and denote $X_{2j} := IX_{2j-1}$ for $j \in \{2, \ldots, n\}$. Denote $(X_1(s), X_2(s), \ldots, X_{2n}(s))$ the orthonormal frame at $\gamma(s)$ obtained from the frame $(X_1, X_2, \ldots, X_{2n})$ by parallel transport along γ. Then*

$$T_{\gamma(\frac{\pi}{2})}\mathbb{R}P^n = \mathrm{Vect}(X_2(\frac{\pi}{2}), X_3(\frac{\pi}{2}), X_5(\frac{\pi}{2}), \ldots, X_{2n-1}(\frac{\pi}{2})).$$

Remark 2.2 (Covering properties for the spaces Y_k)**.** Consider the map

$$\pi_k : Y_k \to (\mathbb{R}P^n)^{k+1}, \qquad \gamma \mapsto (\gamma(0), \gamma(1), \ldots, \gamma(k)),$$

denote $\Delta_k := \{(x_0, \ldots, x_k) \in (\mathbb{R}P^n)^{k+1} : \exists j,\ x_j = x_{j+1}\} \subset (\mathbb{R}P^n)^{k+1}$, let Δ_k^c be the complement of Δ_k and denote $\dot Y_k := \pi_k^{-1}(\Delta_k^c)$. Then $\pi_k : \dot Y_k \to \Delta_k^c$ is a $2^k : 1$ cover and is a $2 : 1$ cover near $K_k \subset \dot Y_k$. The second claim follows from the fact that, given a point x with antipode x', the fiber $\pi_k^{-1}(x, x', x, \ldots)$ contains exactly two geodesics.

3. The homology of $\mathcal{P}_n$

Let X be a Hilbert manifold and $f : X \to \mathbb{R}$ a C^2 function bounded below and satisfying condition (C) of Palais and Smale [20, p. 26]. The function f is called *Morse (or Morse-Bott)* if all its critical points are nondegenerate (or, respectively, if they all lie on nondegenerate critical submanifolds). If $a \in \mathbb{R}$, let

$$X^{\leq a},\ X^{<a},\ X^{=a}$$

denote the sublevel and level sets $\{f \leq a\}$, $\{f < a\}$, respectively $\{f = a\}$. A Morse function f is *perfect for coefficients in a field* $\mathbb{K}$ if the number of critical points of index k is equal to the rank of $H_k(X;\mathbb{K})$ for all $k \in \mathbb{Z}$. This is equivalent to the vanishing of the connecting homomorphisms $H_\cdot(X^{\leq a}, X^{<a};\mathbb{K}) \to H_{\cdot-1}(X^{<a};\mathbb{K})$ at each critical value a, and also to the fact that $H_\cdot(X;\mathbb{K})$ is isomorphic to the direct sum of the level homology groups $H_\cdot(X^{\leq a}, X^{<a};\mathbb{K})$ at the critical levels a. Roughly speaking, f is considered "perfect" if it has the minimal critical set required by the topology of its domain, i.e. by $H_\cdot(X;\mathbb{K})$. For a perfect Morse function the link given by Morse theory between the homology of X and the critical points of f takes the simplest possible form. We will call a Morse-Bott function f *perfect for coefficients in a field* $\mathbb{K}$ if $H_\cdot(X;\mathbb{K})$ is isomorphic to the direct sum of the level homology groups $H_\cdot(X^{\leq a}, X^{<a};\mathbb{K})$ at the critical levels a.

In the literature the term "perfect Morse function" usually means that the isomorphism between the homology of the total space and the direct sum of the level homologies holds with coefficients in any field [5].

Theorem 3.1. *The norm functional on $\mathcal{P}_n$ for the Fubini-Study metric on $\mathbb{C}P^n$ is a* perfect Morse function for $\mathbb{Z}/2$-coefficients*, i.e. the homology of $\mathcal{P}_n$ with $\mathbb{Z}/2$-coefficients is the direct sum of the* level homology groups *of F:*

$$H_\cdot(\mathcal{P}_n;\mathbb{Z}/2) \simeq H_\cdot(\mathbb{R}P^n;\mathbb{Z}/2) \oplus \bigoplus_{k\geq 1} H_\cdot(\mathcal{P}_n^{\leq k\pi/2}, \mathcal{P}_n^{<k\pi/2};\mathbb{Z}/2). \tag{8}$$

If n is odd, the norm functional is a strong perfect Morse function*, meaning that equation* (8) *holds with arbitrary coefficients.*

The theorem implies of course that the energy functional E is also a perfect Morse function with $\mathbb{Z}/2$-coefficients, and a strong perfect Morse function if n is odd.

Remark 3.1. If n is even, we do not know whether F is a strong perfect Morse function or not.

The proof of the theorem relies on the fact that the Y_k's are completing manifolds in the sense of Definition 3.1 below. The latter is reminiscent of [5, p. 531] and [18, p. 97] (see also [23, IX.7], [10, p. 979]).

For X and f as above, let

$$K := \mathrm{Crit}(f) \cap f^{-1}(0) \subset X$$

be the critical locus of f at level 0 and assume K is a Morse-Bott nondegenerate manifold of index $\iota(K)$.

Definition 3.1. Let R be a ring. Let Y be a closed manifold, $L \subset Y$ a submanifold of codimension $\iota(K)$, and let $\varphi : Y \to X^{\leq 0}$. Then (Y, L, φ) is a *local completing manifold for K* if it satisfies condition (i), and a *completing manifold for K with R-coefficients* if it satisfies conditions (i) and (ii) below.

(i) the map φ is an embedding *near* L, it maps L diffeomorphically onto K, and
$$\varphi^{-1}(K) = L.$$
(ii) the canonical map
$$H_\cdot(Y;R) \to H_\cdot(Y, Y \setminus L; R)$$
is surjective.

Remark 3.2. By an arbitrarily small perturbation of φ along $-\nabla f$ we will have pushed all noncritical points at level 0 below level 0, so that we obtain a map
$$\widetilde{\varphi} : (Y, Y \setminus L) \to (X^{\leq 0}, X^{<0})$$
satisfying the same conditions as φ. We can, and shall, assume without loss of generality that $\widetilde{\varphi} = \varphi$.

Lemma 3.2. *Let (Y, L, φ) be a local completing manifold equipped with a retraction $p : Y \to L$ and denote the inclusion $s : L \hookrightarrow Y$. Let o_Y be the orientation local system of Y with $\mathbb{Z}$-coefficients, and let R be any ring. If*
$$p^*s^*(o_Y \otimes R) = o_Y \otimes R, \tag{9}$$
then (Y, L, φ) is a completing manifold with R-coefficients.

Proof. In this proof all local systems are understood to have fiber R and all homology groups are understood to be defined with R-coefficients. Let $\mathrm{incl} : Y \hookrightarrow (Y, L)$ be the inclusion, denote ν the normal bundle to L in Y and let o_ν be the orientation local system for ν. This is a local system on Y whose monodromy along a loop is minus the identity iff the loop is orientation reversing for ν. Then we have a commutative diagram
$$\begin{tikzcd} H_\cdot(Y) \arrow[r, "\mathrm{incl}_*"] \arrow[dr, "s_!"'] & H_\cdot(Y, Y \setminus L) \arrow[d, "\simeq"', "Thom"] \\ & H_{\cdot - \mathrm{codim}(L)}(L; o_\nu) \end{tikzcd} \tag{10}$$
Here the vertical arrow is the Thom isomorphism (composed with excision) and $s_!$ is given by $PD \circ s^* \circ PD$, with PD denoting Poincaré duality.

Let us now consider the retraction $p : Y \to L$, which satisfies by definition the relation $ps = \mathrm{Id}_L$. The maps $s_!$ and $p_!$ act as
$$\begin{tikzcd} H_\cdot(Y) \arrow[rr, "s_!"] & & H_{\cdot - \mathrm{codim}(L)}(L; o_\nu) \arrow[dll, "p_!"'] \arrow[d, dotted, "\mathrm{Id}"] \\ H_\cdot(Y; (p^*s^*o_Y) \otimes o_Y) \arrow[rr, "s_!"'] & & H_{\cdot - \mathrm{codim}(L)}(L; o_\nu) \end{tikzcd}$$

and the composition $s_! p_!$ in this diagram is the identity. In particular the map $s_! : H_\cdot(Y; (p^* s^* o_Y) \otimes o_Y) \to H_{\cdot - \mathrm{codim}(L)}(L; o_\nu)$ is surjective. Our assumption (9) implies that the groups $H_\cdot(Y)$ and $H_\cdot(Y; (p^* s^* o_Y) \otimes o_Y)$ are equal. Therefore the map $s_! : H_\cdot(Y) \to H_{\cdot - \mathrm{codim}(L)}(L; o_\nu)$ is surjective as well, and so is incl_*. □

Corollary 3.3. *Let (Y, L, φ) be a local completing manifold equipped with a retraction $p : Y \to L$. If Y is orientable, then (Y, L, φ) is a completing manifold with any choice of coefficients. If Y is non-orientable, then (Y, L, φ) is a completing manifold with $\mathbb{Z}/2$-coefficients.* □

Lemma 3.4. *Let (Y, L, φ) with $\varphi : Y \to X^{\le 0}$ be local completing manifold data for $K = \mathrm{Crit}(f) \cap f^{-1}(0)$ as in Definition 3.1. Let ι be the index of K.*

(a) the map φ induces a canonical morphism

$$\varphi_* : H_\cdot(Y, Y \setminus L) \to H_\cdot(X^{\le 0}, X^{<0}).$$

(b) Given a submanifold $Z \subset Y$ that is transverse to L, denote $B := Z \cap L$ and let k be the dimension of B. The codimension of B in Z is ι, and the class

$$\varphi_*([Z, Z \setminus B]) \in H_{k+\iota}(X^{\le 0}, X^{<0})$$

is the image under the Thom isomorphism $H_k(K; o_{\nu^-}) \xrightarrow{\sim} H_{k+\iota}(X^{\le 0}, X^{<0})$ of the class

$$\varphi_*[B] \in H_k(K; o_{\nu^-}).$$

Proof. (a) The morphism φ_* is well-defined in view of Remark 3.2. More precisely, one considers a perturbation of φ of the form $\tilde{\varphi}_\varepsilon := \phi^\varepsilon_{-\nabla F} \circ \varphi$ where $\phi^\varepsilon_{-\nabla F}$, $\varepsilon > 0$ is the time-ε flow of $-\nabla F$. All the maps $\tilde{\varphi}_\epsilon$ are homotopic and act as $(Y, Y \setminus L) \to (X^{\le 0}, X^{<0})$.

(b) The key point is identifying the normal bundle. Transversality implies that the normal bundle to B in Z is isomorphic to the restriction of the pull-back bundle $\varphi^* \nu^-$ to B, and the conclusion follows. □

Lemma 3.5. *Let X, f, and K be as for Definition 3.1, denote ν^- the negative bundle of K (of rank $\iota(K)$), and let R be a coefficient ring. If K admits a completing manifold with R-coefficients then we have short exact sequences in homology with R-coefficients*

$$0 \to H_\cdot(X^{<0}) \to H_\cdot(X^{\le 0}) \to H_{\cdot - \iota(K)}(K; o_{\nu^-}) \to 0.$$

If K admits an orientable local completing manifold equipped with a retraction, these exact sequences are naturally split for any choice of coefficients, so that

$$H_\cdot(X^{\le 0}) \simeq H_\cdot(X^{<0}) \oplus H_{\cdot - \iota(K)}(K; o_{\nu^-}).$$

If K admits a non-orientable local completing manifold equipped with a retraction, these exact sequences are naturally split with $\mathbb{Z}/2$-coefficients.

Proof. Let (Y, L, φ) be the completing manifold data for K. In view of Remark 3.2 we can assume without loss of generality that φ acts as

$$\varphi : (Y, Y \setminus L) \to (X^{\le 0}, X^{<0}).$$

By functoriality of the long exact sequence of a pair we obtain a commutative diagram

$$\begin{tikzcd} H.(X^{<0}) \arrow[r] & H.(X^{\leq 0}) \arrow[r, "(\mathrm{incl}_X)_*"] & H.(X^{\leq 0}, X^{<0}) \\ H.(Y \setminus L) \arrow[r] \arrow[u] & H.(Y) \arrow[r, two heads, "\mathrm{incl}_*"] \arrow[u, "\varphi_*"'] & H.(Y, Y \setminus L) \arrow[u, "\simeq"', "\varphi_*^{rel}"] \end{tikzcd}$$

The map incl_* is surjective by assumption. That the map φ_*^{rel} is an isomorphism follows from the fact that φ is an embedding near L, maps L diffeomorphically onto the nondegenerate critical manifold K, and maps $Y \setminus L$ into $X^{<0}$, so that

$$(\varphi|_L)^* \nu^- \simeq \nu.$$

Thus φ_*^{rel} can be written as the composition

$$H.(Y, Y \setminus L) \simeq H._{-\iota(K)}(L; o_\nu) \simeq H._{-\iota(K)}(K; o_{\nu^-}) \simeq H.(X^{\leq 0}, X^{<0}),$$

where the first map is the Thom isomorphism composed with (the inverse of) excision, and the third map follows from Morse-Bott theory [8]. Thus $(\mathrm{incl}_X)_*$ is surjective, which implies the exactness of the short sequence in the statement.

If Y is a local completing manifold with retraction p, we invoke Lemma 3.2, and restrict to $\mathbb{Z}/2$-coefficients if Y is not orientable. We then obtain a section σ_X of $(\mathrm{incl}_X)_*$ given by the composition $\sigma_X := \varphi_* \circ \sigma \circ (\varphi_*^{rel})^{-1} : H.(X^{\leq 0}, X^{<0}) \to H.(X^{\leq 0})$, where $\sigma : H.(Y, Y \setminus L) \to H.(Y)$ is the section of incl_* constructed in the proof of Lemma 3.2 from the map $p_!$.

Given the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H.(X^{<0}) \arrow[r] & H.(X^{\leq 0}) \arrow[r, shift right, "(\mathrm{incl}_X)_*"'] & H.(X^{\leq 0}, X^{<0}) \arrow[l, shift right, "\sigma_X"'] \arrow[r] & 0 \end{tikzcd}$$

we have $H.(X^{\leq 0}) = H.(X^{<0}) \oplus \sigma_X(H.(X^{\leq 0}, X^{<0}))$. Since σ_X is an isomorphism onto its image and since $H.(X^{\leq 0}, X^{<0}) \simeq H._{-\iota(K)}(K; o_{\nu^-})$, we obtain the isomorphism in the statement. □

In practice, the distinction between completing manifolds and completing manifolds equipped with a retraction is null if one restricts to field coefficients. In this situation all short exact sequences split (though not naturally) and thus the direct sum decomposition in Lemma 3.5 is automatic.

Proof of Theorem 3.1. Recall the map $\varphi_k : Y_k \to \mathcal{P}_n$ and the submanifold $L_k \subset Y_k$ from (1) and (2). We now show that the triple (Y_k, L_k, φ_k) is a local completing manifold for K_k equipped with a retraction $Y_k \to L_k$, from which the Theorem follows in view of Lemma 3.5. Indeed, if n is odd, the manifolds Y_k are orientable since they are obtained as fiber products of orientable manifolds over evaluation maps to orientable manifolds.

The map φ_k correctly sends Y_k into $\mathcal{P}_n^{\leq k\pi/2}$. To check Condition (i) in Definition 3.1, let us first note that the map φ_k is an embedding outside the locus of points in Y_k for which at least one of the circles C_{x_j,v_j,θ_j} is constant, and in particular φ_k is an embedding near L_k. Secondly, the map φ_k sends by definition L_k diffeomorphically onto K_k. Finally, it follows from the definition of φ_k that $\varphi_k^{-1}(\mathcal{P}_n^{=k\pi/2})$ consists of k-tuples of great half-circles of length $\pi/2$ with matching endpoints, and therefore the only critical points at level $k\pi/2$ lie in L_k.

To construct a retraction we consider the map

$$p_k : Y_k \to SN\mathbb{R}P^n, \qquad \big((x_0,v_0,\theta_0),\ldots,(x_{k-1},v_{k-1},\theta_{k-1})\big) \longmapsto (x_0,v_0).$$

Via the identifications $SN\mathbb{R}P^n \equiv K_k \equiv L_k$ we can view this map as $p_k : Y_k \to L_k$. Its composition with the inclusion $L_k \hookrightarrow Y_k$ is clearly the identity map of L_k, hence p_k is a retraction of Y_k onto L_k. □

Remark 3.3 (Fiber bundle structure on Y_k)**.** The map $p_k : Y_k \to SN\mathbb{R}P^n$ actually defines a fiber bundle structure on Y_k. To see this we note that p_k can be written as a composition

$$Y_k \to Y_{k-1} \to \ldots \to Y_1 \to SN\mathbb{R}P^n.$$

Here $Y_j \to Y_{j-1}$, $2 \leq j \leq k$ is the projection on the first $(j-1)$-components of any j-tuple. This defines a fiber bundle structure on Y_j with base Y_{j-1} and fiber $S^{n-1} \times S^1_\pi$, where S^{n-1} is the sphere of radius 1 in $\mathbb{R}^n$. The map $Y_1 \to SN\mathbb{R}P^n$ is the projection on the first component, recalling that $Y_1 = SN\mathbb{R}P^n \times S^1_\pi$. As a consequence, p_k is a fiber bundle with fiber $(S^{n-1})^{k-1} \times (S^1_\pi)^k$.

The following statement is a rephrasing of Theorem 3.1.

Corollary 3.6. *For $n \geq 1$ we have an isomorphism of graded $\mathbb{Z}/2$-modules*

$$H_\cdot(\mathcal{P}_n;\mathbb{Z}/2) \simeq H_\cdot(\mathbb{R}P^n;\mathbb{Z}/2) \oplus \bigoplus_{k\geq 1} H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}/2)[-1-(k-1)n] \tag{11}$$

If n is odd, denote $o_{\nu_k^-}$, $k \geq 1$ the local system of orientations for the negative bundle to K_k and view this as a local system on $ST\mathbb{R}P^n$ via the identifications $ST\mathbb{R}P^n \equiv SN\mathbb{R}P^n \equiv K_k$. We have an isomorphism of graded $\mathbb{Z}$-modules

$$H_\cdot(\mathcal{P}_n) \simeq H_\cdot(\mathbb{R}P^n) \oplus \bigoplus_{k\geq 1} H_\cdot(ST\mathbb{R}P^n; o_{\nu_k^-})[-1-(k-1)n]. \tag{12}$$

□

We will now compute these graded $\mathbb{Z}$-modules explicitly if n is odd. Along the way, we will also determine the local systems $o_{\nu_k^-}$ for the case of even n.

Lemma 3.7.

(a.1) If n is odd, the local system $o_{\nu_k^-}$ is trivial for all $k \geq 1$.

(a.2) If n is even, the local system $o_{\nu_k^-}$ is trivial for odd $k \geq 1$, and it is nontrivial for even $k \geq 2$. In the second case we have

$$o_{\nu_k^-} = \pi^* o, \tag{13}$$

where o is the orientation local system on $\mathbb{R}P^n$.

(b) If n is odd, there is an isomorphism

$$H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}) = H_\cdot(\mathbb{R}P^n;\mathbb{Z}) \otimes H_\cdot(S^{n-1};\mathbb{Z}). \tag{14}$$

Proof. To prove (b), let n be odd and consider the Leray-Serre spectral sequence with $\mathbb{Z}$-coefficients for the bundle $S^{n-1} \hookrightarrow ST\mathbb{R}P^n \xrightarrow{\pi} \mathbb{R}P^n$. The second page is given by $H_\cdot(\mathbb{R}P^n;\mathbb{Z}) \otimes H_\cdot(S^{n-1};\mathbb{Z})$ and $d^n = 0$ since the Euler class of $\mathbb{R}P^n$ is zero. The differentials d^r, $r \neq n$ vanish for dimension reasons. Recalling that the integral homology of $\mathbb{R}P^n$ in (ascending) degree $\leq n$ is $H_\cdot(\mathbb{R}P^n;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}/2, 0, \ldots, \mathbb{Z}/2, 0, \mathbb{Z})$ and using that $Ext(\mathbb{Z}, \mathbb{Z}/2) = 0$ we obtain (14).

A similar argument allows one to compute $H_\cdot(ST\mathbb{R}P^n;\mathbb{Z})$ for n even. The second page of the spectral sequence is $H_\cdot(\mathbb{R}P^n;\mathbb{Z}) \oplus H_\cdot(\mathbb{R}P^n;o)[-n+1]$. The integral homology of $\mathbb{R}P^n$ in (ascending) degree $\leq n$ is $H_\cdot(\mathbb{R}P^n;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}/2, 0, \ldots, \mathbb{Z}/2, 0)$, and the homology with coefficients in o in (ascending) degree $\leq n$ is isomorphic to $(\mathbb{Z}/2, 0, \ldots, \mathbb{Z}/2, 0, \mathbb{Z})$. Here we use the Poincaré duality isomorphism $H_\cdot(\mathbb{R}P^n;o) \simeq H^{n-\cdot}(\mathbb{R}P^n;\mathbb{Z})$. The differential d^n is thus 0 since all its components have either vanishing source or vanishing target, whereas the differentials d^r, $r \neq n$ vanish for dimension reasons. This determines unambiguously the homology $H_\cdot(ST\mathbb{R}P^n;\mathbb{Z})$ in all degrees except $n-1$, where we know that it is a $\mathbb{Z}/2$-extension of $\mathbb{Z}/2$. By writing the spectral sequence with $\mathbb{Z}/2$-coefficients and using that the Euler class of $\mathbb{R}P^n$ is equal to 1, we obtain that $H_{n-1}(ST\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2$, and thus $H_{n-1}(ST\mathbb{R}P^n;\mathbb{Z}) = \mathbb{Z}/4$.

In particular, we obtain that the fundamental group of $ST\mathbb{R}P^n$ is

$$\pi_1(ST\mathbb{R}P^n) = \left\{ \begin{array}{ll} \mathbb{Z}/4 & \text{if } \ n = 2 \\ \mathbb{Z}/2 & \text{if } \ n \geq 3 \end{array} \right\}.$$

Indeed, the fundamental group is abelian since $ST\mathbb{R}P^n$ is a $\mathbb{Z}/2$-quotient of STS^n, which has cyclic fundamental group, and is therefore isomorphic to $H_1(ST\mathbb{R}P^n;\mathbb{Z})$.

We now discuss nontrivial local systems on $ST\mathbb{R}P^n$. Note that $\mathbb{R}P^n$, $n \geq 1$ carries a unique nontrivial local system, denoted τ, because its fundamental group has a single generator (of even order). If n is even, $\mathbb{R}P^n$ is not orientable and thus $\tau = o$. When n is odd, o is trivial. While on $ST\mathbb{R}P^1 = S^1 \sqcup S^1$ there are two nontrivial local systems (modulo diffeomorphisms), on $ST\mathbb{R}P^n$, $n \geq 2$ there is a unique nontrivial local system since the fundamental group still has a single generator which has even order. The map π_* sends a generator of the fundamental group of the total space of the bundle to a generator of the fundamental group of the base. Thus the unique notrivial local system on $ST\mathbb{R}P^n$, $n \geq 2$ is equal to $\pi^*\tau$.

We now prove statements *(a.1)* and *(a.2)*. Note that (13) is a direct consequence of the fact that the unique nontrivial local system on $ST\mathbb{R}P^n$, $n \geq 2$ is $\pi^*\tau$, and $\tau = o$ if n is even. Recall for the proof that we have denoted $L_k \subset Y_k$ the diffeomorphic image of K_k under the embedding (2). We view $o_{\nu_k^-}$ as the local system of orientations for the normal bundle ν_k to L_k in Y_k and denote it as such by o_{ν_k}. We identify L_k with $ST\mathbb{R}P^n$ as above.

The manifold $\mathbb{R}P^n$ is orientable if and only if n is odd, but $ST\mathbb{R}P^n$ is always orientable. Let $\widetilde{Y}_k$ be an open neighborhood of L_k in Y_k that retracts onto L_k. Then o_{ν_k} is trivial if and only if $\widetilde{Y}_k$ is orientable, and it is sufficient to check the value of the orientation system of $\widetilde{Y}_k$ on the image of one of the generators of $\pi_1(ST\mathbb{R}P^n)$. We will use the generator represented by the tangent vector to the loop (in homogeneous coordinates) $\gamma(t) = [\cos t : \sin t : 0 : ... : 0]$, $0 \leq t \leq \pi$ in $\mathbb{R}P^n$. Recall that $\widetilde{Y}_k$ is parametrized by sequences $(x_0, ..., x_k) \in (\mathbb{R}P^n)^{k+1}$ where the x_i are the endpoints of the k half-circles making up a path in the image of $\widetilde{Y}_k$. For each $t \in [0, \pi]$ the element of L_k corresponding to the tangent vector $\dot{\gamma}(t) \in ST\mathbb{R}P^n$ is parametrized by the sequence

$$(x_0, ..., x_k)(t) = (\gamma(t), \dot{\gamma}(t), \gamma(t), \dot{\gamma}(t), ...) \in (\mathbb{R}P^n)^{k+1}.$$

Note that in each case the path $x_i(t)$ represents a generator of $\pi_1\mathbb{R}P^n$. The value of the orientation system $o^{\otimes k+1}$ on $(\mathbb{R}P^n)^{k+1}$ on this loop in $ST\mathbb{R}P^n$ is -1 if and only if n and k are even. Thus we have *(a.1)* and *(a.2)*. □

Remark 3.4. Here is an alternative proof of statements *(a.1)* and *(a.2)*. We use the global involution

$$A : Y_{k+1} \to Y_{k+1}$$

described heuristically as reversing the time direction on the path represented by an element of Y_{k+1}. The involution A is a diffeomorphism which acts by sending a $(k+1)$-tuple $\big((x_j, v_j, \theta_j)\big)_{0\leq j\leq k}$ to $\big((x'_j, v'_j, \theta'_j)\big)_{0\leq j\leq k}$, with (x'_j, v'_j) the *opposite* of the endpoint speed vector of $C_{x_{k-j}, v_{k-j}, \theta_{k-j}}$ and $\theta_j := -\theta_{k-j}$. Our convention is that the endpoint speed vector of a constant circle $C_{x_{k-j}, v_{k-j}, 0}$ is $(x_{k-j}, -v_{k-j})$, so that the endpoint speed vector defines a smooth function on Y_1. In particular $(x'_j, v'_j) = (x_{k-j}, v_{k-j})$. Denoting by ev_0, $\mathrm{ev}_1 : Y_{k+1} \to \mathbb{R}P^n$ the evaluation maps at the endpoints, we have

$$\mathrm{ev}_1 = \mathrm{ev}_0 \circ A.$$

To prove *(a.1)* and *(a.2)* we think of L_k as being identified with $SN\mathbb{R}P^n$ as above. We also recall the notation ν_k for the normal bundle to L_k in Y_k. Since $Y_1 = SN\mathbb{R}P^n \times S^1_\pi$ we obtain that ν_1 is trivial, and so is o_{ν_1}.

Let now $k \geq 2$. Write $Y_k = Y_{k-1}\,{}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_0} Y_1$ and embed $Y_{k-1} \hookrightarrow Y_k$ by adding to any $(k-1)$-tuple $\big((x_j, v_j, \theta_j)\big)_{0\leq j\leq k-2}$ the geodesic half-circle $(x_{k-1}, v_{k-1}, \pi/2)$, where (x_{k-1}, v_{k-1}) is the endpoint speed vector of $C_{x_{k-2}, v_{k-2}, \theta_{k-2}}$. This embedding is a section for the fiber bundle $Y_k \to Y_{k-1}$. The normal bundle to Y_{k-1} in Y_k under this embedding then satisfies

$$\nu_{Y_k} Y_{k-1} \oplus \mathbb{R} = (\mathrm{ev}_1)^*(N\mathbb{R}P^n \oplus \mathbb{R}).$$

In this equality we think of the trivial rank one factor on the left hand side as being generated by the section v_{k-1} along Y_{k-1}, and use the identification between $N_{x_{k-1}}\mathbb{R}P^n$ and $\langle v_{k-1}\rangle \oplus T_{v_{k-1}} SN_{x_{k-1}}\mathbb{R}P^n$. The trivial rank one factor on the right hand side corresponds to varying the argument $\theta_{k-1} \in S^1_\pi$. We obtain at the level of orientation local systems

$$o_{\nu_{Y_k} Y_{k-1}} = (\mathrm{ev}_1)^* o_{N\mathbb{R}P^n} = (\mathrm{ev}_1)^* o_{T\mathbb{R}P^n}.$$

Since $\mathrm{ev}_1 = \mathrm{ev}_0 \circ A$ and since we only consider local systems up to diffeomorphisms, we can alternatively write

$$o_{\nu_{Y_k} Y_{k-1}} = (\mathrm{ev}_0)^* o_{T\mathbb{R}P^n}.$$

By induction we then obtain

$$o_{\nu_k} = (\mathrm{ev}_0)^* (o_{T\mathbb{R}P^n})^{\otimes k-1}.$$

Note that $\mathrm{ev}_0 = \pi$ on Y_1. Assertions *(a.1)* and *(a.2)* then follow from the fact that $o_{T\mathbb{R}P^n}$ is trivial if n is odd (orientable case), respectively nontrivial if n is even (nonorientable case).

4. The Pontryagin-Chas-Sullivan product

In this section we restrict to homology with $\mathbb{Z}/2$-coefficients. Our motivation is that the Pontryagin-Chas-Sullivan product extends the intersection form on the homology of $\mathbb{R}P^n$, and the latter is most natural with $\mathbb{Z}/2$-coefficients. Our purpose is to prove Theorem 1.1, which is stated at the beginning of the paper.

By formula (11) we have

$$H_\cdot(\mathcal{P}_n;\mathbb{Z}/2) \simeq H_\cdot(\mathbb{R}P^n;\mathbb{Z}/2) \oplus \bigoplus_{k\geq 1} H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}/2)[-1-(k-1)n]. \qquad (15)$$

The second page of the spectral sequence for computing $H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}/2)$ simplifies to $H_\cdot(\mathbb{R}P^n;\mathbb{Z}/2) \otimes H_\cdot(S^{n-1};\mathbb{Z}/2)$, i.e.

$$\begin{tikzcd}
n-1 & \mathbb{Z}/2 & \mathbb{Z}/2 & \dots & \mathbb{Z}/2 & \mathbb{Z}/2 \\
0 & \mathbb{Z}/2 & \mathbb{Z}/2 & \dots & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[ullll, "\cdot e"'] \\
 & 0 & 1 & \dots & n-1 & n
\end{tikzcd}$$

The only depicted differential is multiplication by the Euler number e modulo 2, which vanishes for n odd and is an isomorphism for n even. Thus for n odd

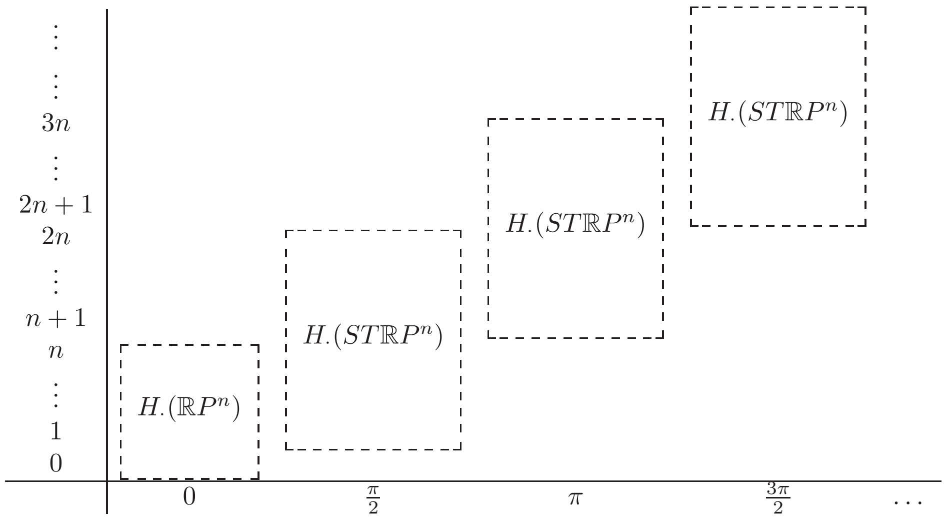
$$H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}/2) = H_\cdot(\mathbb{R}P^n;\mathbb{Z}/2) \otimes H_\cdot(S^{n-1};\mathbb{Z}/2),$$

whereas for n even

$$H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{in degrees lying in } \{0,\dots,2n-1\}, \\ 0, & \text{else.} \end{cases}$$

Convention. In the rest of this section we shall not include anymore the coefficient ring $\mathbb{Z}/2$ in the notation for homology groups. As an example, we shall write $H_\cdot(ST\mathbb{R}P^n)$ instead of $H_\cdot(ST\mathbb{R}P^n;\mathbb{Z}/2)$.

It is convenient to write the homology $H_\cdot(\mathcal{P}_n)$ as given by Theorem 3.1 and formula (15) in a table in which the homological degree appears as the vertical coordinate, and the critical values of the norm F appear as the horizontal coordinate.



4.1. Generalities.

Our computation of the multiplicative structure of the ring $(H_\cdot(\mathcal{P}_n), *)$ makes use of several general principles which are of larger interest and which we now emphasize. Whereas the discussion in (1–3) below is valid for arbitrary path spaces, item (4) is specific to the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$.

(1) Geometric realization of product classes.

For the next Lemma we recall the codimension n submanifold $\mathcal{C}_n \subset \mathcal{P}_n \times \mathcal{P}_n$ and the concatenation map $c : \mathcal{C}_n \to \mathcal{P}_n$.

Lemma 4.1. *Let $[\chi], [\zeta] \in H_\cdot(\mathcal{P}_n)$ be two classes which are respectively represented by C^1-maps $\chi : M \to \mathcal{P}_n$, $\zeta : N \to \mathcal{P}_n$ whose sources are closed finite-dimensional manifolds M, N. Assume that the maps*

$$\mathrm{ev}_1 \circ \chi : M \to \mathbb{R}P^n, \qquad \mathrm{ev}_0 \circ \zeta : N \to \mathbb{R}P^n$$

are transverse. Denote their fiber product by

$$C := M \ {}_{\mathrm{ev}_1\circ\chi}\!\times_{\mathrm{ev}_0\circ\zeta} \ N,$$

with the natural map $(\chi, \zeta) : C \to \mathcal{C}_n$. Then

$$[\chi] * [\zeta] = [c \circ (\chi, \zeta)].$$

Proof. Recall the Eilenberg-Zilber map $EZ : H_\cdot(\mathcal{P}_n) \otimes H_\cdot(\mathcal{P}_n) \to H_\cdot(\mathcal{P}_n \times \mathcal{P}_n)$, the inclusion $s : \mathcal{C}_n \to \mathcal{P}_n \times \mathcal{P}_n$ and the fact that $[\chi] * [\zeta] = c_* s_! EZ([\chi] \otimes [\zeta])$. The transversality assumption in the statement is equivalent to the fact that the map $\chi \times \zeta : M \times N \to \mathcal{P}_n \times \mathcal{P}_n$ is transverse to $\mathcal{C}_n$. Thus $s_!([\chi \times \zeta]) = [(\chi, \zeta) : C \to \mathcal{C}_n]$. In view of the equality $EZ([\chi] \otimes [\zeta]) = [\chi \times \zeta]$, the Lemma follows. $\square$

Remark 4.1 (Geometric representing cycles for Pontryagin-Chas-Sullivan products in the case of submersive evaluation maps)**.** With the same notation as above, suppose that $\mathrm{ev}_0 \circ \zeta : N \to \mathbb{R}P^n$ is a submersion. Then the hypothesis of Lemma 4.1 is satisfied for *every* C^1-map $\chi : M \to \mathcal{P}_n$, and the Pontryagin-Chas-Sullivan product is represented by the naive formula

$$[\chi] * [\zeta] = [\{\chi_x \cdot \zeta_y \ : \ x \in M,\ y \in N,\ \chi_x(1) = \zeta_y(0)\}], \tag{16}$$

the set on the right being naturally parameterized by the appropriate fiber product.

Here is an example that will be important to us later. Denote $[Y_k] \in H_{(k+1)n}(\mathcal{P}_n)$, $k \geq 1$ the class represented by the "parametrization map" $\varphi_k : Y_k \to \mathcal{P}_n$. Since $\mathrm{ev}_0 \circ \varphi_{k-1} : Y_{k-1} \to \mathbb{R}P^n$ is a submersion, formula (16) is a natural and simple way of right-multiplying by $[Y_k]$ on the chain level. As a consequence we have by induction and using the definition of Y_k:

$$[Y_k] = [Y_1]^{*k}. \tag{17}$$

An analogous discussion holds for left multiplication with $[Y_k]$. Indeed, formula (16) also holds for every C^1-map $\zeta : N \to \mathbb{R}P^n$ provided that $\mathrm{ev}_1 \circ \chi : M \to \mathbb{R}P^n$ is a submersion. In our case, the map $\mathrm{ev}_1 \circ \varphi_k : Y_k \to \mathbb{R}P^n$ is a submersion.

(2) Min-max critical levels.

Let X be a Hilbert manifold and $f : X \to \mathbb{R}$ a function of class C^2 satisfying condition (C) of Palais and Smale. Given a homology class $\alpha \in H.(X)$, the "minimax" *critical level of α with respect to f* is

$$\mathrm{Crit}(\alpha; f) := \min\{c \in \mathbb{R} \ : \ \alpha \in \mathrm{im}\big(H.(\{f \leq c\}) \to H.(X)\big)\}.$$

If there is no danger of confusion, we shall write $\mathrm{Crit}(\alpha)$ instead of $\mathrm{Crit}(\alpha; f)$.

We clearly have

$$\mathrm{Crit}(\beta) \leq \max(\mathrm{Crit}(\alpha), \mathrm{Crit}(\alpha + \beta)). \tag{18}$$

Let $Y \subseteq X$ be a closed subset. We say that Y *supports a homology class $\delta \in H.(X)$* if $\delta \in \mathrm{im}\big(H.(Y) \to H.(X)\big)$. We say that Y *supports a linear subspace $V \subseteq H.(X)$* if it supports every element of V.

Corollary 4.2. *Suppose $H.(X)$ is spanned over $\mathbb{Z}/2$ by subspaces V and W, and let $\delta \in H.(X)$. If there is a compact set $Y \subset X$ supporting V and δ, but no nonzero element of W, then $\delta \in V$.*

Proof. There is a smooth, nonnegative function $\rho : X \to \mathbb{R}$ whose 0-set is Y. Our hypotheses imply that $\mathrm{Crit}(\alpha; \rho) = 0$ for all $\alpha \in V$, and $\mathrm{Crit}(\delta; \rho) = 0$. Let us now write $\delta = \alpha + \beta$ with $\alpha \in V$ and $\beta \in W$. Equation (18) implies $\mathrm{Crit}(\beta; \rho) = 0$, so that $\beta = 0$ and $\delta \in V$. □

Now let $f : X \to \mathbb{R}$ be a function as above and assume in addition that it is Morse-Bott. Let K be a connected component of the critical set of f at level c, with index ι. We allow the critical set of f at level c to be disconnected and the index to vary from one component to the other. For simplicity we use $\mathbb{Z}/2$-coefficients so that the local system of orientations for the negative bundle on K is trivial, but this discussion generalizes in a straightforward way to arbitrary coefficients. Let $\tau : H_{\cdot-\iota}(K) \to H_\cdot(X^{\leq c}, X^{<c})$ be induced from the Thom isomorphism. As an immediate consequence of Corollary 4.2 we have:

Corollary 4.3. *Assume that $H_\cdot(K)$ is spanned over $\mathbb{Z}/2$ by subspaces V and W, and that $\delta \in H_\cdot(K)$. If there is a compact set $Y \subset K$ supporting V and δ, but no nonzero element of W, then $\tau(\delta) \in \tau(V)$.*

In particular, if every nonzero element of W has nonzero intersection product with an element of $H_\cdot(K)$ that can be supported in $K - Y$, then $\tau(\delta) \in \tau(V)$.

We shall not use in the sequel the second part of Corollary 4.3.

Critical levels are well-behaved with respect to the Pontryagin-Chas-Sullivan product provided one uses the *norm*

$$F := \sqrt{E}.$$

More precisely, we have the following

Lemma 4.4. *Let $\alpha, \beta \in H_\cdot(\mathcal{P}_n)$. Min-max critical levels of the norm functional F satisfy the inequality*

$$\mathrm{Crit}(\alpha * \beta; F) \leq \mathrm{Crit}(\alpha; F) + \mathrm{Crit}(\beta; F). \tag{19}$$

□

The proof of (19) given in [16, Proposition 5.3] for the case of the free loop space holds verbatim in our situation. Note that the critical value $\mathrm{Crit}(\alpha; F)$ is equal to 0 for all classes α that are represented by constant paths. Note also that inequality (19) does *not* hold for critical levels of the energy functional E since the latter is not additive with respect to simultaneous concatenation and reparametrization of the paths (see the discussion in [16, §10.6]). This is main advantage of the norm functional F over the more popular energy functional E.

(3) Symmetries.

Let

$$A : \mathcal{P}_n \to \mathcal{P}_n, \qquad (A\gamma)(t) := \gamma(1-t)$$

be the involution given by reversing the direction of paths. (This is consistent with the involution $Y_k \to Y_k$ defined in Remark 3.4.) The involution A fixes the space of constant paths pointwise. The critical set of E is stable under A since E is A-invariant. We will also denote by A the map induced in homology.

Lemma 4.5. *For all $\alpha, \beta \in H_\cdot(\mathcal{P}_n)$ we have*

$$A(\alpha * \beta) = A(\beta) * A(\alpha).$$

Proof. Let T be the self-diffeomorphism of $\mathcal{P}_n \times \mathcal{P}_n$ given by $(\gamma, \delta) \mapsto (\delta, \gamma)$. Then $(A \times A) \circ T = T \circ (A \times A)$ is a self-diffeomorphism of $\mathcal{P}_n \times \mathcal{P}_n$ which preserves $\mathcal{C}_n$ and satisfies

$$c \circ \big((A \times A) \circ T\big) = A \circ c,$$

as well as $\big((A \times A) \circ T\big) \circ s = s \circ \big((A \times A) \circ T\big)$. The result then follows directly from the definition of the $*$-product, using the naturality of the Eilenberg-Zilber map and the fact that it commutes with T (in the obvious sense). $\square$

Remark 4.2. The space $\mathcal{P}_n$ carries yet another involution which we denote $\gamma \mapsto \bar{\gamma}$ and call *complex conjugation*. This is induced by complex conjugation on complex projective space

$$z = [z_0 : z_1 : \cdots : z_n] \mapsto \bar{z} = [\bar{z}_0 : \bar{z}_1 : \cdots : \bar{z}_n],$$

which is an involution that fixes $\mathbb{R}P^n$ pointwise.

The involutions $^{-}$ and A commute with each other. They also commute with the map

$$\mathcal{P}_n \hookrightarrow \mathcal{P}_{n+1}$$

induced by any real linear embedding $(\mathbb{C}P^n, \mathbb{R}P^n) \hookrightarrow (\mathbb{C}P^{n+1}, \mathbb{R}P^{n+1})$. If γ is a vertical half-circle, then so are $\bar{\gamma}$ and $A\gamma$.

(4) Heredity.

As we have already alluded to in the previous paragraph, the embedding

$$(\mathbb{C}P^n, \mathbb{R}P^n) \hookrightarrow (\mathbb{C}P^{n+1}, \mathbb{R}P^{n+1}), \qquad [z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : z_n : 0]$$

induces an embedding

$$\mathcal{P}_n \hookrightarrow \mathcal{P}_{n+1}.$$

We denote the induced map in homology

$$f_n : H_\cdot(\mathcal{P}_n) \to H_\cdot(\mathcal{P}_{n+1}).$$

We emphasize that f_n is *not* a morphism of rings (not only does it fail to send the unit into the unit, but it also does not respect the product structure).

However, the map f_n is *linear*. The following *Heredity Principle* is a simple but useful rephrasing of this fact: given a linear relation between homology classes in $H_\cdot(\mathcal{P}_n)$, their images under f_n must satisfy the *same* linear relation. This principle is effective since our generators of $H_\cdot(\mathcal{P}_n)$ are geometric, so that their images under f_n are easy to identify.

4.2. Hopf symmetries and section (n odd)

Let $n = 2m + 1$. The fibers of the Hopf principal bundle

$$S^1 \to S^n \to \mathbb{C}P^m$$

are (oriented) great circles on S^n. The quotient of the total space by the subgroup $\mathbb{Z}/2 \subset S^1$ is the total space of a principal $S^1/(\mathbb{Z}/2) = \mathbb{R}P^1$-bundle

$$\mathbb{R}P^1 \to \mathbb{R}P^n \to \mathbb{C}P^m.$$

Because the Hopf fibers are geodesics, and S^1 acts by isometries, the fibers of the map $\mathbb{R}P^n \to \mathbb{C}P^m$ are (oriented) geodesics on $\mathbb{R}P^n$.

The map that assigns to $x \in S^n$ the unit tangent vector u to the Hopf fiber at x is a global section of the unit tangent bundle STS^n. If we view STS^n as the set $\{(x,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|x\| = 1,\ \|u\| = 1,\ \langle x,u\rangle = 0\}$ with $\langle\cdot,\cdot\rangle$ the Euclidean scalar product, then this section can be viewed as the map

$$x = (x_0, x_1, \ldots, x_{n-1}, x_n) \mapsto u = (-x_1, x_0, \ldots, -x_n, x_{n-1}).$$

In complex coordinates this is $u = Jx$, where $J \in S^1$ is a fourth root of unity.

If $n \equiv 3 \pmod 4$, then S^n can be identified with the unit sphere in the quaternionic vector space $\mathbb{H}^{(n+1)/4}$. In this case there are three such sections

$$u = J_i x, \qquad i \in \{1,2,3\}$$

so that for each $x \in S^n$ the vectors $Jx = J_1x,\ J_2x,\ J_3x \in T_xS^n$ form an orthonormal triple.

The map that assigns to $x \in \mathbb{R}P^n$ the tangent vector u to the Hopf fiber at x is a global section of the unit tangent bundle $ST\mathbb{R}P^n$ that descends from the above section of STS^n by viewing $ST\mathbb{R}P^n$ as the quotient of STS^n by the action of the derivative of the antipodal map $S^n \to S^n$. The expression $u = Jx$ also makes sense for $x \in \mathbb{R}P^n$ and $u \in ST_x\mathbb{R}P^n$, and describes the section

$$\begin{aligned} \sigma : \mathbb{R}P^n &\to ST\mathbb{R}P^n, \\ x &\mapsto u = Jx. \end{aligned} \tag{20}$$

It follows that the Hopf fiber through x in $\mathbb{R}P^n$ is the unique geodesic with initial vector $u = Jx$. Now let I be the complex structure on $T\mathbb{C}P^n$. Then

$$x \mapsto v = IJx$$

describes a section of the unit *normal* bundle $SN\mathbb{R}P^n$ to $\mathbb{R}P^n$ in $\mathbb{C}P^n$.

The direction $v = IJx$ at $x \in \mathbb{R}P^n \subset \mathbb{C}P^n$ determines the complex line $\ell_{x,v}$, whose intersection with $\mathbb{R}P^n$ is thus precisely the Hopf fiber through x. Note also that the equator of $\ell_{x,v}$ inherits an orientation from the Hopf fiber, and that $\ell_{x,v}$ has a globally defined "upper hemisphere", see Figure 4.

Define

$$S_1 := \{(x,v,\theta) \in Y_1 : v = IJx\} \ \subseteq\ Y_1 = SN\mathbb{R}P^n \times S^1.$$

Note that

$$S_1 = \{\text{vertical half circles } \gamma : \gamma \subset \ell^+_{\gamma(0)}\}$$

where $\ell^+_{\gamma(0)}$ is the upper half of the complex line containing the Hopf fiber of $\gamma(0)$. Note also that for each vertical half circle γ we have $\gamma \subset \ell^+_{\gamma(0)}$ if and only if $A\gamma \subset \ell^+_{\gamma(1)}$. This gives us the first statement in the next lemma.

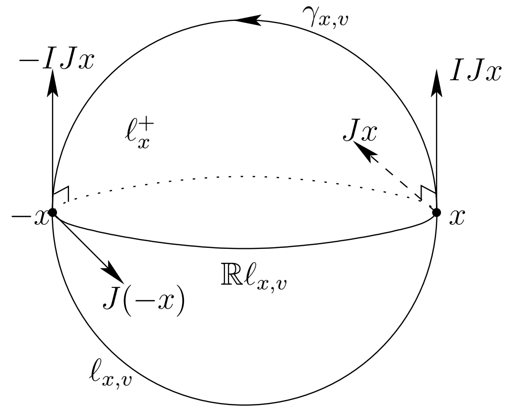


FIGURE 4. Hopf fibers in S^{2m+1}.

Lemma 4.6. *We have*

$$AC(S_1) = C(S_1),$$
$$AC(Y_1) = C(Y_1),$$

where $C : Y_1 \to \mathcal{P}_n$ *is the parametrization map*

$$(x, v, \theta) \mapsto C_{x,v,\theta}.$$

Proof. The statement is a consequence of the fact that if γ is a vertical half circle, so is $A\gamma$. See the example $n = 1$ worked out below. □

4.3. Generators

Let us first describe the generators of $H_.(\mathcal{P}_n)$ with respect to the $*$-product for $n \geq 1$.

- the "constant loops class" $U \in H_n(\mathcal{P}_n)$. This is the image of the fundamental class $[\mathbb{R}P^n]$ under the injection $H_.(\mathbb{R}P^n) \hookrightarrow H_.(\mathcal{P}_n)$. It is represented by the submanifold $\mathbb{R}P^n \subset \mathcal{P}_n$ consisting of constant loops.
- the "hyperplane class" $H \in H_{n-1}(\mathcal{P}_n)$. It is represented by the class of a hyperplane $\mathbb{R}P^{n-1}$ in the constant loops $\mathbb{R}P^n \subset \mathcal{P}_n$.
- the "completing manifold class" $Y \in H_{2n}(\mathcal{P}_n)$. This is represented by the map

$$C : Y_1 \to \mathcal{P}_n,$$
$$(x, v, \theta) \mapsto C_{x,v,\theta}.$$

- the "class of a completed section", $S \in H_{n+1}(\mathcal{P}_n)$ for $n = 2m+1$ odd. This is represented by the map
$$C : S_1 \to \mathcal{P}_n,$$
$$(x, v, \theta) \mapsto C_{x,v,\theta}.$$
- the "class of a completed section defined along a hyperplane", $T \in H_n(\mathcal{P}_n)$ for $n = 2m$ even. We choose a hyperplane $\mathbb{R}P^{2m-1} \subset \mathbb{R}P^{2m}$ and consider again the section $\mathbb{R}P^{2m-1} \to ST\mathbb{R}P^{2m-1}$ defined above, viewed as taking values in $ST\mathbb{R}P^{2m}$. The class T is represented by the restriction of C to
$$\{(x, v, \theta) \in Y_1 : x \in \mathbb{R}P^{2m-1} \text{ and } v = IJx\}$$
(For n even, J is only defined along the hyperplane $\mathbb{R}P^{2m-1}$.)
  Alternatively, the class $T \in H_n(\mathcal{P}_n)$, $n = 2m$ even is the image of S in $H_n(\mathcal{P}_{n-1})$ under the map $H_n(\mathcal{P}_{n-1}) \to H_n(\mathcal{P}_n)$.

It follows from Lemma 4.6 and the Heredity Principle in §4.1(4) that
$$AS = S, \qquad AY = Y, \qquad AH = H, \qquad AT = T \tag{21}$$
We denote $P \in H_0(\mathcal{P}_n)$ the class of a point (taken to be a constant loop, for example).

Remark 4.3 (The case $n = 1$). When $n = 1$ we have that $H = P$. In this case $ST\mathbb{R}P^1$ is a disjoint union of two circles and the completing manifold is $Y_1 = ST\mathbb{R}P^1 \times S^1_\pi$, which is a disjoint union of two tori. For every $x = [x_0 : x_1] \in \mathbb{R}P^1$, the vector $Jx = [-x_1 : x_0]$ lies in $T_x\mathbb{R}P^1$, and IJx (defined using the vector Jx and the complex structure I on $\mathbb{C}P^1$) is tangent to $\mathbb{C}P^1$ but normal to $\mathbb{R}P^1$. The vectors IJx with $x \in \mathbb{R}P^1$ point into the "upper" hemisphere of $\mathbb{C}P^1$ (the hemisphere about which $\mathbb{R}P^1$ has positive winding number) and the vectors $-IJx$ with $x \in \mathbb{R}P^1$ point into the "lower" hemisphere. (Complex conjugation reverses the two hemispheres and fixes the equator $\mathbb{R}P^1$.) Thus *S is represented by the space of vertical half circles in the upper hemisphere with endpoints on $\mathbb{R}P^1$, parametrized by the endpoints $\{(x, x') \in \mathbb{R}P^1 \times \mathbb{R}P^1\}$.* The class $Y \in H_2(\mathcal{P}_1)$ is represented by the sum of S and its complex conjugate $\overline{S}$ in the lower hemisphere:
$$Y = S + \overline{S}.$$

Using our previous computation of $H_\cdot(\mathcal{P}_n)$, we are now in a position to write down explicitly generators of all the homology groups. For simplicity we shall sometimes suppress the symbol $*$ from the multiplicative notation, and write for example HY instead of $H * Y$. For readability we refer to a critical level $k\pi/2$ as being "level k". We recall that $\mathbb{R}P^n$ denotes the subset of constant paths inside $\mathcal{P}_n$, which coincides also with the critical level set at 0. Since the restriction of the Pontryagin-Chas-Sullivan product to the trivial paths is the intersection product, we have that $H_\cdot(\mathbb{R}P^n) = \mathrm{Vect}(U, H, H^2, \dots, H^n)$. Here $\mathrm{Vect}(\mathcal{S})$ denotes the (graded) vector space generated by a (graded) set $\mathcal{S}$.

We have seen in Theorem 3.1 that we have an isomorphism with suitable coefficients
$$H_\cdot(\mathcal{P}_n) \simeq H_\cdot(\mathbb{R}P^n) \oplus \bigoplus_{k \geq 1} H_\cdot(\mathcal{P}_n^{\leq k\pi/2}, \mathcal{P}_n^{< k\pi/2}).$$

The completing manifold (Y_k, L_k, φ_k), $k \geq 1$ with retraction

$$p_k : Y_k \to L_k \simeq SN\mathbb{R}P^n$$

given in the proof of Theorem 3.1 induces splitting maps

$$\sigma_k : H.(Y_k, Y_k \backslash L_k) \to H.(Y_k)$$

for the long exact sequence of the pair $(Y_k, Y_k \setminus L_k)$. These in turn induce splitting maps

$$\widetilde{\sigma}_k : H.(\mathcal{P}_n^{\leq k\pi/2}, \mathcal{P}_n^{<k\pi/2}) \to H.(\mathcal{P}_n^{\leq k\pi/2})$$

for the long exact sequence of the pair $(\mathcal{P}_n^{\leq k\pi/2}, \mathcal{P}_n^{<k\pi/2})$. For each $k \geq 1$ there is a commutative diagram

$$\begin{tikzcd}
H.(Y_k, Y_k \setminus L_k) \arrow[r, "\simeq"] \arrow[d, shift left, "\sigma_k"] & H.(\mathcal{P}_n^{\leq k\pi/2}, \mathcal{P}_n^{<k\pi/2}) \arrow[d, shift left, "\tilde{\sigma}_k"] & \\
H.(Y_k) \arrow[u, shift left] \arrow[r] \arrow[rr, bend right, "(\varphi_k)_*"'] & H.(\mathcal{P}_n^{\leq k\pi/2}) \arrow[u, shift left] \arrow[r, "i_k"] & H.(\mathcal{P}_n)
\end{tikzcd}$$

with the map i_k being determined by inclusion, the other two horizontal arrows being determined by φ_k, and the bottom-to-top vertical arrows being induced by inclusion. We refer to the *block on level k (in the homology of $H.(\mathcal{P}_n)$)* as being $\mathrm{im}((\varphi_k)_* \circ \sigma_k) = \mathrm{im}(i_k \circ \tilde{\sigma}_k)$. We have seen that the block on level k is isomorphic to the homology $H.(ST\mathbb{R}P^n)[-1-(k-1)n]$, and we shall sometimes refer by abuse of language to the block on level k as being this last homology group.

Lemma 4.7. *Multiplication on the right by $Y^{s-\ell}$ determines an isomorphism between the blocks on levels $k = \ell$ and $k = s$ for $s \geq \ell \geq 1$ and $n \geq 2$.*

(i) Let n odd ≥ 3. The block on level 1 is

$$H.(ST\mathbb{R}P^n)[-1] \simeq H.(\mathbb{R}P^n) * \mathrm{Vect}(S, Y).$$

(ii) Let n even ≥ 2. The block on level 1 is

$$H.(ST\mathbb{R}P^n)[-1] \simeq \mathrm{Vect}(U, H, H^2, \ldots, H^{n-1}) * \mathrm{Vect}(T, Y).$$

(iii) Let $n = 1$. The block $H.(ST\mathbb{R}P^1)[-k]$ on level k is isomorphic to $H.(\mathbb{R}P^1) * \mathrm{Vect}(S\bar{S}S\ldots, \bar{S}S\bar{S}\ldots)$, where each of the products $\ldots S\bar{S}S\ldots$ has k factors.

In loose terms the content of the first part of Lemma 4.7 is that the essential piece of information is contained on the level 1 block: the latter is replicated infinitely many times, with suitable – monotonically diverging – degree shifts, via multiplication with powers of the class Y. The second part of Lemma 4.7 computes the homology of the level 1 block. The resulting explicit description of a set of generators for the homology groups $H.(\mathcal{P}_n)$ is the following.

- **n even ≥ 2.** The block $H_\cdot(\mathbb{R}P^n)$ on level 0 is generated by $U, H, H^2, \ldots, H^n = P$. The block $H_\cdot(ST\mathbb{R}P^n)[-1-(k-1)n]$ on level k is generated by $TY^{k-1}, HTY^{k-1}, \ldots, H^{n-1}TY^{k-1}$ in (descending) degrees $kn, \ldots, 1+(k-1)n$, and $Y^k, HY^k, \ldots, H^{n-1}Y^k$ in (descending) degrees $(k+1)n, \ldots, 1+kn$.
- **n odd ≥ 3.** The block $H_\cdot(\mathbb{R}P^n)$ on level 0 is generated by $U, H, H^2, \ldots, H^n = P$. The block $H_\cdot(ST\mathbb{R}P^n)[-1-(k-1)n]$ on level k has generators $SY^{k-1}, HSY^{k-1}, H^2SY^{k-1}, \ldots, H^nSY^{k-1}$ in (descending) degrees $1+kn, \ldots, 1+(k-1)n$, and $Y^k, HY^k, H^2Y^k, \ldots, H^nY^k$ in descending degrees $(k+1)n, \ldots, kn$.
- **$\mathbf{n = 1}$.** The block $H_\cdot(\mathbb{R}P^1)$ on level 0 is generated by U (in degree 1) and $H = P$ (in degree 0). The block $H_\cdot(ST\mathbb{R}P^1)[-k] = H_\cdot(S^1 \sqcup S^1)[-k]$ on level k is generated by $S\overline{S}S\ldots$ and $\overline{S}S\overline{S}\ldots$($k$ factors each) in degree $k+1$, and by $HS\overline{S}S\ldots$ and $H\overline{S}S\overline{S}\ldots$in degree k. (Note that according to our conventions, $Y = S + \overline{S}$.)

It is useful to depict the first levels for the cases $n = 1, 2, 3, 4$ in a common table.

	0	$\frac{\pi}{2}$	π	$\ldots$	0	$\frac{\pi}{2}$	$\ldots$	0	$\frac{\pi}{2}$	$\ldots$	0	$\frac{\pi}{2}$	$\ldots$
9													
8												Y	
7												HY	
6									Y			H^2Y	
5									HY			H^3Y	
4						Y			S H^2Y		U_4	T	
3			$S\overline{S}, \overline{S}S$			HY		U_3	HS H^3Y		H	HT	
2		$S, \overline{S}$	$HS\overline{S}, H\overline{S}S$		U_2	T		H	H^2S		H^2	H^2T	
1	U_1	$HS,\ H\overline{S}$			H	HT		H^2	H^3S		H^3	H^3T	
0	H				H^2			H^3			H^4		
		$n = 1$				$n = 2$			$n = 3$			$n = 4$	

Remark 4.4 (More heredity properties)**.** The maps $\mathcal{P}_n \to \mathcal{P}_{n+1}$, $n \geq 2$ induce injections in homology on the first two columns of the above table. In particular

$$H^k \mapsto H^{k+1},\ H^kT \mapsto H^{k+2}S,\ H^kS \mapsto H^kT.$$

Note that the degree of the generators on the other columns goes uniformly to ∞ as $n \to \infty$. This yields in particular the computation of the homology of the space

$$\mathcal{P}_\infty := \mathcal{P}_{\mathbb{R}P^\infty}\mathbb{C}P^\infty := \varinjlim_n \mathcal{P}_n.$$

The homology $H_\cdot(\mathcal{P}_\infty)$ is the limit of the first two columns in the above table as $n \to \infty$, it has rank 1 in degree 0 and rank 2 in all the other degrees.

Proof of Lemma 4.7. The statement concerning right multiplication by $Y^{s-\ell}$ follows from Lemma 4.1, which relates the fiber product of two manifolds to the Pontryagin-Chas-Sullivan product of the homology classes they represent. To see this, we recall the identification $SN\mathbb{R}P^n \equiv ST\mathbb{R}P^n$, the bundle maps $p_k : Y_k \to ST\mathbb{R}P^n$, $k \geq 1$ described in Remark 3.3, and the parametrization maps $\varphi_k : Y_k \to \mathcal{P}_n$, $k \geq 1$ in §3. Also, given a closed submanifold $X \subset Y_k$ we denote $[X] \in H_\cdot(\mathcal{P}_n)$ the class represented by the map $\varphi_k|_X$.

Let now $B \subset ST\mathbb{R}P^n$ be a closed submanifold and denote $B_k := (p_k)^{-1}(B) \subset Y_k$, $k \geq 1$. Then

$$B_s = B_\ell \ {}_{\mathrm{ev}_1 \circ \varphi_\ell}\!\times_{\mathrm{ev}_0 \circ \varphi_{s-\ell}} Y_{s-\ell}$$

by definition of the manifolds Y_k, $k \geq 1$. By associativity of the map $c_{\min}$ we have $c_{\min} \circ (\varphi_\ell, \varphi_{s-\ell}) = \varphi_s$, and Lemma 4.1 implies

$$[B_s] = [B_\ell] * [Y_{s-\ell}].$$

By (17) we obtain $[B_s] = [B_\ell] * Y^{s-\ell}$. Since for each $k \geq 1$ the classes $[B_k]$ form a basis of the block on level k as B ranges through a collection of submanifolds of $ST\mathbb{R}P^n$ which represent a basis of $H_\cdot(ST\mathbb{R}P^n)$, the conclusion follows.

To prove statement (i), let us recall the computation of $H_\cdot(ST\mathbb{R}P^n; \mathbb{Z}/2)$ in §3. A basis of $H_\cdot(ST\mathbb{R}P^n)$ is obtained by restricting the section σ in (20), respectively the bundle $ST\mathbb{R}P^n$, to the linear subspaces $\mathbb{R}P^k$, $0 \leq k \leq n$. Note in particular that $H_\cdot(ST\mathbb{R}P^n)$ has a basis consisting of classes that are represented by submanifolds.

To understand the block $H_\cdot(ST\mathbb{R}P^n)[-1]$ on level 1, recall that Y_1 is a (trivial) S^1_π-bundle over $K_1 \equiv ST\mathbb{R}P^n$ by $(x, v, \theta) \overset{p_1}{\mapsto} (x, v)$. Each cycle in $ST\mathbb{R}P^n$ gives rise to a cycle in $\mathcal{P}_n$ by taking its preimage under the bundle map p_1 and mapping into $\mathcal{P}_n$ via φ_1. In view of the above description of $H_\cdot(ST\mathbb{R}P^n)$, and in view of the definition of the classes S and Y, we obtain $H_\cdot(ST\mathbb{R}P^n)[-1] \simeq H_\cdot(\mathbb{R}P^n) * \mathrm{Vect}(S, Y)$. Indeed, the classes in $H_\cdot(\mathbb{R}P^n)$ are represented by constant loops and performing a Pontryagin-Chas-Sullivan product of a homology class $\alpha \in H_\cdot(\mathcal{P}_n)$ with such constant loop classes amounts to restricting the locus of the starting point of the paths in a representing cycle for α.

The proof of statement (ii) is similar to the proof of statement (i).

Statement (iii) follows from the above and from the discussion in Remark 4.3. $\square$

Remark 4.5 (On representing cycles for homology classes)**.** For convenience, we have chosen to phrase Lemma 4.7 in terms of of homology. Nevertheless, one could also state it in terms of cycles; the outcome is a set of very simple and explicit representatives for a basis of the $\mathbb{Z}/2$-vector space $H_\cdot(\mathcal{P}_n)$. First we have geometric cycles representing the level 0 classes U, H, $H^2, \ldots, H^n$ in $H_\cdot(\mathcal{P}_n^{=0})$, and the level 1 classes S, T, Y in $H_\cdot(\mathcal{P}_n^{\leq \pi/2})$, which are defined at the beginning of §4.3 by geometric representatives. As we have just discussed, the block on level 1 is generated by S (or T if n is even), Y, and the classes $H^j S$ (respectively $H^j T$), and $H^j Y$, $1 \leq j \leq n$ represented by "restrictions" of our representatives for S, T, Y to projective subspaces of lower dimension. Right-multiplication by Y^k maps our list of generators at level 1 isomorphically onto a set of generators at level $k+1$ and, as discussed in Remark 4.1, the right-multiplication of our generators by Y^k can be realized at chain level as a fiber-product with $\mathrm{ev}_0 \circ \varphi_k : Y_k \to \mathbb{R}P^n$ (see formulas (16) and (17)).

One should be able to define right- (or left-) multiplication by Y^k as a chain map; elaborating upon this point in full generality definitely goes beyond the scope of the present paper.

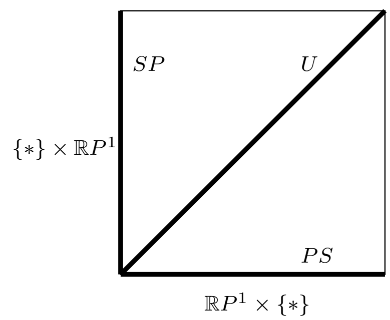


FIGURE 5. The space of half-circles contained in the upper hemisphere on $\mathbb{C}P^1$ is modeled on $\mathbb{R}P^1 \times \mathbb{R}P^1$ using the endpoints.

4.4. Relations

At this point we have already proved half of our main theorem, since we have seen that $1, H, S, Y$, respectively $1, H, T, Y$ generate $H_\cdot(\mathcal{P}_n)$ as a ring. We now have to identify the relations satisfied by these generators.

4.4.1. The case $n = 1$.

Lemma 4.8. *The following relations hold for $n = 1$:*

$$PS + SP = P\overline{S} + \overline{S}P = U, \qquad S^2 = 0, \qquad \overline{S}^2 = 0.$$

Proof. To prove the relation $S^2 = 0$ we note that S^2 is represented by a cycle in $\mathcal{P}_1^{\leq\pi}$ and this cycle does not contain any geodesics of length π. Hence $\mathrm{Crit}(S^2) < \pi$. On the other hand S^2 lives in degree 3, but $H_3(\mathcal{P}_1^{<\pi}) = 0$ since $H_\cdot(\mathcal{P}_1^{<\pi})$ is a vector space spanned by P, U_1, HS, $H\overline{S}$, S, and $\overline{S}$ (see the previous table). Thus S^2 must vanish. The relation $\overline{S}^2 = 0$ is proved similarly.

We now prove $PS + SP = U$. The Hopf fibration determines an orientation on $\mathbb{R}P^1$, and at any point $x \in \mathbb{R}P^1$, the vector $v = IJx$ points into the "upper hemisphere" ℓ^+ of $\mathbb{C}P^1$. Then S can be alternatively described as the class of the cycle $s : \mathbb{R}P^1 \times \mathbb{R}P^1 \to \mathcal{P}_1$ that associates to a pair of points (x, x') the unique vertical half-circle from x to x' and contained in ℓ^+. The cycle PS is represented by the restriction of s to $\{*\} \times \mathbb{R}P^1$, and SP is represented by the restriction of s to $\mathbb{R}P^1 \times \{*\}$. If we represent the torus $\mathbb{R}P^1 \times \mathbb{R}P^1$ as a square with opposite sides identified, the cycles PS and SP correspond to two adjacent sides. (See Figure 5.) The cycle $U = U_1$ in turn corresponds to the diagonal, hence $PS + SP = U$.

The relation $P\overline{S} + \overline{S}P = U$ can be proved in the same way. □

Proof of Theorem 1.1 in the case $n = 1$. We have already noticed above that

$$Y = S + \overline{S}, \qquad H = P.$$

The homology of $\mathcal{P}_1$ on level $\pi/2$ can alternatively be described as

$$H_.(ST\mathbb{R}P^1)[-1] = \mathrm{Vect}(HS, S, HY, Y),$$

and more generally the homology of $\mathcal{P}_1$ on level $k\pi/2$ can be described as

$$H_.(ST\mathbb{R}P^1)[-k] = \mathrm{Vect}(HS, S, HY, Y) * Y^{k-1} = \mathrm{Vect}(HSY^{k-1}, SY^{k-1}, HY^k, Y^k).$$

The relations in Lemma (4.8) imply

$$HS + SH = 1, \qquad HY + YH = 0, \qquad S^2 = 0,$$

as well as

$$YS + SY = Y^2.$$

Finally, we have $H^2 = 0$ since H is the class of a constant path. $\square$

4.4.2. The case $n \geq 3$ odd.

Lemma 4.9. *Let $n \geq 3$ odd. The following relation holds*

$$HS + SH = U.$$

Proof. For index reasons, SH is a linear combination of U, HS, H^nY. We first show that SH is a linear combination of U and HS. The image of the section $\sigma : \mathbb{R}P^n \hookrightarrow ST\mathbb{R}P^n = K_1$ is a closed set supporting homology classes whose image under the Thom isomorphism $\tau : H_.(K_1) \to H_{.+1}(\mathcal{P}_n^{\leq \frac{\pi}{2}}, \mathcal{P}_n^{< \frac{\pi}{2}})$ contains HS and SH (both of which are represented by "subsets of S") but not supporting H^nY. Indeed, the image of σ is diffeomorphic to $\mathbb{R}P^n$, the rank of $H_n(\mathbb{R}P^n)$ is equal to 1, but HS and H^nY span a 2-dimensional subspace of the homology. This is exactly the situation covered by Corollary 4.3 and we infer that the class SH is a multiple of HS mod $H_.(\mathcal{P}_n^{< \frac{\pi}{2}})$.

We also have:

- $SH \neq 0$ by Lemma 4.5 (since $HS \neq 0$) .
- $SH \neq HS$ (otherwise we would obtain $SH^n = H^nS$, whereas Lemma 4.8 and Remark 4.4 on heredity prove that $SH^n + H^nS = H^{n-1}$).
- $SH \neq U$ (otherwise $U = A(U) = A(SH) = HS$).

Thus $SH = HS + U$, which is equivalent to $SH + HS = U$. $\square$

Lemma 4.10. *Let $n \geq 1$ odd. The following relation holds*

$$S + \overline{S} = \begin{cases} 0, & \text{if } n \equiv 3 \pmod 4, \\ H^{n-1}Y, & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Proof. We have already seen that this is true if $n = 1$. By the second assertion in Lemma 3.5 on the homological consequences of the existence of a completing manifold, and because there is no homology in $\mathcal{P}_n$ in the degree of S below level $\frac{\pi}{2}$, it is enough to prove the corresponding relation (under the Thom isomorphism) in the homology of $SN\mathbb{R}P^n \simeq ST\mathbb{R}P^n$. Let us denote by $h^k s$ and $h^k y$ the generators in $H.(ST\mathbb{R}P^n)$ corresponding to $H^k S$ and $H^k Y$. They are represented by

$$h^k s_1 := \{(x,u) \in ST\mathbb{R}P^n : x \in \mathbb{R}P^{n-k} \text{ and } u = Jx\}$$
$$h^k y_1 := \{(x,u) \in ST\mathbb{R}P^n : x \in \mathbb{R}P^{n-k}\}$$

The involution on $ST\mathbb{R}P^n$ corresponding to the involution $\gamma \to \overline{\gamma}$ on $\mathcal{P}_n$ is the map $(x,u) \to (x,-u)$. Thus the generator $\overline{s}$ in $H.(ST\mathbb{R}P^n)$ corresponding to $\overline{S}$ is represented by

$$\overline{s}_1 := \{(x,u) \in ST\mathbb{R}P^n : x \in \mathbb{R}P^n \text{ and } u = -Jx\}$$

The group $H_n(ST\mathbb{R}P^n)$ is generated by the elements s and $h^{n-1}y$. The dual group $H_{n-1}(ST\mathbb{R}P^n)$ is generated by hs and $h^n y$. The reader can verify that

$$\begin{aligned} h^{n-1}y \cap h^n y &= 0, \\ h^{n-1}y \cap hs &= *, \\ s \cap h^n y = \overline{s} \cap h^n y &= *, \\ \overline{s} \cap hs &= 0. \end{aligned}$$

where $(*)$ is the generator of $H_0(ST\mathbb{R}P^n)$. It remains to compute $s \cap hs$.

Let $n \equiv 3 \pmod 4$. In this case the one-parameter family of sections $S^n \to STS^n$ given by $x \mapsto (x, u_t)$ with

$$u_t := (\cos tJ_1 + \sin tJ_2)x$$

descends to a family of sections σ_t: $\mathbb{R}P^n \to ST\mathbb{R}P^n$ that represent s but do not intersect s_1 for $t > 0$. Thus

$$s \cap hs = 0,$$

so that s and $\overline{s}$ have the same intersection numbers and are therefore homologous.

Let now $n \equiv 1 \pmod 4$. We claim that

$$s \cap hs = *, \tag{22}$$

so that the classes s and $\overline{s} + h^{n-1}y$ have the same intersection numbers and are therefore equal.

The sphere S^n is now the unit sphere in $\mathbb{H}^{(n-1)/4} \oplus \mathbb{C}$. On $\mathbb{H}^{(n-1)/4}$ we have the complex structures J_1 and J_2, while on $\mathbb{C}$ we have the complex structure $J = J_1$.

Notation. Given $x \in \mathbb{R}^{n+1}$, we write

$$x =: (y,z) =: (y(x), z(x)),$$

with first component $y = (y_0, ..., y_{n-2}) = (x_0, ..., x_{n-2}) \in \mathbb{R}^{n-1} \simeq \mathbb{H}^{(n-1)/4}$ and second component $z = (z_0, z_1) = (x_{n-1}, x_n) \in \mathbb{R}^2 \simeq \mathbb{C}$.

The one-parameter family of sections $S^n \to STS^n$ given by $(y,z) \to ((y,z), u_t)$ with

$$u_t := ((\cos(t)J_1 + \sin(t)J_2)y, Jz)$$

descends to a family of sections $\sigma_t : \mathbb{R}P^n \to ST\mathbb{R}P^n$. Each of these sections represents s and, if $t > 0$, σ_t agrees with $\sigma = \sigma_0$ precisely when $y = 0$, i.e. when $x \in \mathbb{R}P^1$

which is given by $\{[0:0:...:0:y_{n-1}:y_n]\}$. In particular, each section σ_t intersects $hs_1 = \{(x,u) \in ST\mathbb{R}P^n : y_n = 0 \text{ and } u = Jx\}$ in exactly one point, namely the image in $ST\mathbb{R}P^n$ of the point (x_*, u_*) with

$$x_* = (0,0,....,0,1,0),$$
$$u_* = Jx = (0,0,....,0,1).$$

To prove (22) it is enough to show that this intersection is transverse for some fixed $t > 0$. This is a local question for which we need to verify that the tangent space $T_{(x_*,u_*)}STS^n$ is spanned by the tangent space at (x_*, u_*) to the submanifolds

$$\mathcal{V} := \{(x,u) \in STS^n : x \in S^{n-1} = \{y_n = 0\} \text{ and } u = Jx\}$$

(whose image in $ST\mathbb{R}P^n$ is hs_1) and

$$\mathcal{W} := \{(x, u_t(x)) : x \in S^n\}$$

(whose image represents s).

We view STS^n as

$$STS^n = \{(x,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,:\, \|x\| = 1,\ \|u\| = 1,\ \langle x, u\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product, so that

$$T_{(x,u)}STS^n = \{(\xi,\eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,:\, \langle \xi, x\rangle = 0,\ \langle \eta, u\rangle = 0,\ \langle \xi, u\rangle + \langle x, \eta\rangle = 0\}.$$

Thus

$$T_{(x_*,u_*)}STS^n = \{(\xi,\eta) \,:\, \xi_{n-1} = 0,\ \eta_n = 0,\ \xi_n + \eta_{n-1} = 0\},$$

whereas

$$\begin{aligned} T_{(x_*,u_*)}\mathcal{V} &= \{(\xi,\eta) \in T_{(x_*,u_*)}STS^n \,:\, \xi_n = 0,\ \eta = J\xi\} \\ &= \{(\xi,\eta) \,:\, z(\xi) = z(\eta) = 0,\ y(\eta) = J_1 y(\xi)\} \end{aligned}$$

and

$$T_{(x_*,u_*)}\mathcal{W} = \{(\xi,\eta) \in T_{(x_*,u_*)}STS^n \,:\, y(\eta) = (\cos tJ_1 + \sin tJ_2)y(\xi)\}.$$

For $t \notin \pi\mathbb{Z}$ we see that the intersection of the last two subspaces is $\{0\}$, hence transverse because they have complementary dimensions. □

Lemma 4.11. *Let $n \geq 3$ odd. The following relation holds*

$$YS + SY = \begin{cases} 0, & \text{if } n \equiv 3 \pmod 4, \\ H^{n-1}Y^2, & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Proof. It is enough to prove that

$$YS = \overline{S}Y, \tag{23}$$

from which the result follows in view of Lemma 4.10. In turn, it is enough to prove the relation that corresponds to (23) in the level homology group $H_\cdot(\mathcal{P}_n^{\leq\pi}, \mathcal{P}_n^{<\pi})$ since there is no homology in $\mathcal{P}_n$ in the degree of YS and $\overline{S}Y$ below level π (compare with the proof of Lemma 4.10).

The classes YS and $\overline{S}Y$ are represented by submanifolds of Y_2 via φ_2 as follows. Recall the submanifold $S_1 = \{(x, v, \theta) \in Y_1 \,:\, v = IJx\} \subset Y_1$ defined in §4.2, denote $\overline{S}_1 := \{(x, v, \theta) \in Y_1 \,:\, v = -IJx\} \subset Y_1$, and also recall the map $C : Y_1 \to \mathcal{P}_n$ in §4.2, so that $S = C_*[S_1]$ and $\overline{S} = C_*[\overline{S}_1]$. For YS a representing submanifold is

$$\mathcal{V} := \{((x_0, v_0, \theta_0), (x_1, v_1, \theta_1)) \in Y_2 \,:\, (x_1, v_1, \theta_1) \in S_1\} \subset Y_2,$$

whereas for $\overline{S}Y$ a representing submanifold is

$$\mathcal{W} := \{((x_0, v_0, \theta_0), (x_1, v_1, \theta_1)) \in Y_2 \,:\, (x_0, v_0, \theta_0) \in \overline{S}_1\} \subset Y_2.$$

We then have $\mathcal{V} \cap L_2 = \mathcal{W} \cap L_2$. Indeed, for a pair of geodesic vertical half-circles C_{x_0,v_0,θ_0} and C_{x_1,v_1,θ_1} of length $\pi/2$ whose concatenation is a geodesic of length π, we have $v_0 = -IJx_0$ if and only if $v_1 = IJx_1$. Moreover, the intersection of $\mathcal{V}$ and $\mathcal{W}$ with L_2 is transverse. To see this, we first note that both submanifolds $\mathcal{V}$ and $\mathcal{W}$ have codimension $n-1$. In the case of $\mathcal{V}$, let $\mathbf{p} := ((x_0, v_0, \frac{\pi}{2}), (x_1, v_1, \frac{\pi}{2})) \in \mathcal{V}$ correspond via φ_2 to a geodesic of length π. Denote $\gamma := C_{x_0,v_0,\frac{\pi}{2}}$, $\delta := C_{x_1,v_1,\frac{\pi}{2}}$, and $c(\gamma, \delta)$ their concatenation. The tangent vectors to L_2 that correspond to variations of the tangent vector $\dot{\gamma}(1) = \dot{\delta}(0)$ inside $ST_{\gamma_p(\frac{\pi}{2})}\mathbb{R}P^n$ while keeping the midpoint of the geodesic fixed, span a subspace of dimension $n-1$ that is complementary to $T_{\mathbf{p}}\mathcal{V}$. This proves that the intersection $\mathcal{V} \cap L_2$ is transverse at any point $\mathbf{p}$. The same argument applies to $\mathcal{W}$.

Now Lemma 3.4(b), which describes the behavior of homology classes that are represented by transverse intersections with the critical locus, shows that the submanifolds $\mathcal{V}$ and $\mathcal{W}$ represent the same classes in the level homology group $H_\cdot(\mathcal{P}_n^{\leq \pi}, \mathcal{P}_n^{<\pi})$, hence in $H_\cdot(\mathcal{P}_n)$. Thus $YS = \overline{S}Y$. □

Lemma 4.12. *Let $n \geq 3$ odd. The following relation holds*

$$S^2 = 0.$$

Proof. Denote $I := [0, 1]$ and let

$$\widetilde{S_1} = \{\gamma : (I, \partial I) \to (\mathbb{C}P^n, \mathbb{R}P^n) : \gamma \subset \ell^+_{\gamma(0)}\}.$$

There is a strong deformation retraction $\widetilde{S_1} \to S_1$ that preserves the space of paths in $\widetilde{S_1}$ beginning at x and ending at x' for each pair (x, x') with $x \in \mathbb{R}P^n$ and $x' \in \mathbb{R}\ell_x$ (the Hopf fiber of x). This can be accomplished as follows: there is a natural continuous family of diffeomorphisms

$$\chi_x : D^2 \xrightarrow{\cong} \ell^+_x, \qquad x \in \mathbb{R}P^n,$$

where D^2 is a disk of radius 1 with the flat metric. If x and x' belong to the same Hopf fiber then $\chi_x^{-1}\chi_{x'}$ is an isometry of D^2. We then use affine interpolation, with the inherited affine structure from D^2, between an arbitrary path $\gamma : [0, 1] \to \ell^+_{\gamma(0)}$ and the vertical half circle in $\ell^+_{\gamma(0)}$ with the same endpoints. (This works because D^2 is convex.) Thus the inclusion $S_1 \hookrightarrow \widetilde{S_1}$ is a homotopy equivalence. In particular $H_\cdot(\widetilde{S_1}) = 0$ in degrees greater than $\dim S_1 = n+1$. On the other hand, in view of Lemma 4.1 the degree $n+2$ homology class S^2 is represented by

$$S_1 {}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_0} S_1 := \{\gamma \cdot \delta : \gamma, \delta \in S_1 \text{ and } \gamma(1) = \delta(0)\}$$

since the self-intersection is transverse over the evaluation map. (Here we identify S_1 with its image in $\mathcal{P}_n$.) Clearly $S_1 {}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_0} S_1 \subset \widetilde{S_1}$. Thus the class $S^2 \in H_{n+2}(\mathcal{P}_n)$ is in the image of $H_{n+2}(\widetilde{S_1})$ and therefore vanishes. □

Lemma 4.13. *For all $n \geq 2$ we have*

$$HY = YH.$$

Proof. Since $A(YH) = HY$ we obtain $YH \neq 0$. Now $\mathrm{Crit}(YH) \leq \mathrm{Crit}(Y) + \mathrm{Crit}(H) = \pi/2$. On the other hand YH lives in degree $2n-1$, and the only non-zero class in this degree and critical level $\leq \pi/2$ is HY. Thus $YH = HY$. □

This proves Theorem 1.1 in the case n odd.

4.4.3. The case $n \geq 2$ even.

Lemma 4.14. *Let $n \geq 2$ even. The following relation holds*

$$TY + YT = Y.$$

Proof. For index reasons we have $YT = aTY + bY$ with $a, b \in \mathbb{Z}/2$.

Recall from §4.1(4) the inclusion $\mathcal{P}_n \hookrightarrow \mathcal{P}_{n+1}$ determined by a real hyperplane embedding $(\mathbb{C}P^n, \mathbb{R}P^n) \hookrightarrow (\mathbb{C}P^{n+1}, \mathbb{R}P^{n+1})$ and the induced map $f_n : H_.(\mathcal{P}_n) \to H_.(\mathcal{P}_{n+1})$ in homology. It follows from the geometric description of our cycles that

$$f_n T = H^2 S, \qquad f_n Y = HYH = H^2 Y,$$

$$f_n(TY) = H^2 SYH = H^2 SHY = H^3 SY + H^2 Y,$$

$$f_n(YT) = HYH^2S = H^3YS = \begin{cases} H^3SY, & n+1 \equiv 3 \pmod 4 \\ H^3SY + H^{n+2}Y^2, & n+1 \equiv 1 \pmod 4 \end{cases} = H^3SY.$$

Thus $H^3SY = f_n(YT) = af_n(TY) + bf_nY = aH^3SY + (a+b)H^2Y$. Since H^3SY and H^2Y are linearly independent, this implies $a = 1$ and $b = 1$. □

Lemma 4.15. *Let $n \geq 2$ even. The following relation holds*

$$TH + HT = H.$$

Proof. Consider the map $f_{n-1} : H_.(\mathcal{P}_{n-1}) \to H_.(\mathcal{P}_n)$ induced in homology by the real inclusion $\mathcal{P}_{n-1} \hookrightarrow \mathcal{P}_n$. We have

$$f_{n-1}S = T, \qquad f_{n-1}(HS) = HT, \qquad f_{n-1}U = H.$$

We have $f_{n-1} \circ A = A \circ f_{n-1}$. In view of the relations $AH = H$, $AS = S$, $AT = T$ from (21), and in view of Lemma 4.5 which expresses the compatibility between A and

the Pontryagin-Chas-Sullivan product, we also have $A(HS) = SH$ and $A(HT) = TH$. We thus obtain
$$f_{n-1}(SH) = TH.$$
Now the *linear* relation $SH + HS = U$ implies $TH + HT = H$. (This is an instance of the *Heredity principle* in §4.1(4).) $\square$

Lemma 4.16. *Let $n \geq 2$ even. The following relation holds*
$$T^2 = T.$$

Proof. For readability we divide the proof in several steps.

Preliminaries.

We recall the map $C : Y_1 \to \mathcal{P}_n$, $(x, v, \theta) \mapsto C_{x,v,\theta}$. In order to simplify the notation we write in the sequel γ instead of $C(\gamma)$, with $\gamma = (x, v, \theta) \in \mathcal{P}_n$. Whereas γ refers to an element of Y_1 or to the path $C(\gamma)$ will be clear from the context. We also introduce the notation $\mathbb{P}(V)$ for the projective space associated to a real vector space V. We write $n = 2t$.

We also recall that, given an element $x \in \mathbb{R}P^{2t-1}$, we denote ℓ_x^+ the "upper hemisphere" of the complex line $\ell_{x,Jx}$, i.e. the hemisphere towards which points the vector IJx in $\mathbb{T}_x\mathbb{C}P^{2t-1}$. The boundary ℓ_x^+ is the real projective line through x and tangent to Jx, i.e. the fiber of the Hopf fibration $\mathbb{R}P^{2t-1} \to \mathbb{C}P^{t-1}$ obtained by factorizing the standard Hopf fibration $S^{2t-1} \xrightarrow{2:1} \mathbb{R}P^{2t-1} \longrightarrow \mathbb{C}^{t-1}$. We denote this Hopf fiber by $h_x := \partial\ell_x^+$, or $h_x^{\mathbb{R}P^{2n-1}}$ in order to emphasize the total space of the Hopf fibration.

Description of T.

Recall that T is the image of S under the map $f_{n-1} : H_n(\mathcal{P}_{n-1}) \to H_n(\mathcal{P}_n)$ induced by the embedding
$$(\mathbb{C}P^{n-1}, \mathbb{R}P^{n-1}) \hookrightarrow (\mathbb{C}P^n, \mathbb{R}P^n)$$
given by $[z_0 : \cdots : z_{n-1}] \mapsto [z_0 : \cdots : z_{n-1} : 0]$. Thus T is represented by
$$\begin{aligned} T_1 &:= \{\gamma \in Y_1 : \gamma(0) \in \mathbb{R}P^{n-1} \text{ and } \mathrm{im}(\gamma) \subset \ell^+_{\gamma(0)}\} \\ &= \{\gamma \in Y_1 : \gamma(1) \in \mathbb{R}P^{n-1} \text{ and } \mathrm{im}(\gamma) \subset \ell^+_{\gamma(1)}\}. \end{aligned}$$

Let us write $\mathbb{R}P^n = \mathbb{P}(\mathbb{R}^{2t-2} \times \mathbb{R}^3)$ and denote $\mathbb{P}_1 := \mathbb{R}P^{2t-1} = \mathbb{P}(\mathbb{R}^{2t-2} \times \mathbb{R}^2 \times \{0\})$. Then
$$T_1 = \{\gamma \in Y_1 : \gamma(0) \in \mathbb{P}_1,\ \gamma(1) \in h^{\mathbb{P}_1}_{\gamma(0)} \text{ and } \mathrm{im}(\gamma) \subset \ell^+_{\gamma(0)}\}.$$

Yet another description of T_1 is
$$T_1 = \{(x_0, x_2) \in \mathbb{P}_1 \times \mathbb{P}_1 \,:\, x_2 \in h^{\mathbb{P}_1}_{x_0}\}. \tag{24}$$
Indeed, given $(x_0, x_2) \in T_1$ there is a unique vertical half-circle C_{x_0,x_2} contained in the upper half-sphere $\ell^+_{x_0}$ and joining x_0 and x_2. The class T is represented by the map $(x_0, x_2) \mapsto C_{x_0,x_2}$.

Description of T^2.

In order to describe a representing cycle for T^2 we first note that T_1 is not self-transverse along the evaluation map at the endpoints. In order to apply Lemma 4.1 we use the following perturbation of T_1. We denote $\mathbb{P}'_1 := \mathbb{P}(\mathbb{R}^{2t-2} \times \{0\} \times \mathbb{R}^2)$ and identify the product $\mathbb{R}^{2t-2} \times \{0\} \times \mathbb{R}^2$ with $\mathbb{C}^t$ and endow it with a complex structure J such that the subspace $\mathbb{R}^{2t-2} \times \{0\}$ is J-invariant and inherits the same complex structure as from the above identification $\mathbb{R}^{2t-2} \times \mathbb{R}^2 \times \{0\} \equiv \mathbb{C}^t$. We get a Hopf fibration $\mathbb{P}'_1 \to \mathbb{C}P^{t-1}$ with fibers $h_x^{\mathbb{P}'_1}$ and "upper hemispheres" $\ell_x'^{+}$, and we define

$$T'_1 := \{\gamma \in Y_1 : \gamma(0) \in \mathbb{P}'_1,\ \gamma(1) \in h_{\gamma(0)}^{\mathbb{P}'_1} \text{ and } \gamma \subset \ell_{\gamma(0)}'^{+}\}.$$

Since we can interpolate from $\mathbb{R}^2 \times \{0\}$ to $\{0\} \times \mathbb{R}^2$ inside $\mathbb{R}^3$ by a continuous family of planes endowed with complex structures, we infer that T'_1 is homologous to T_1. Since the cycles T'_1 and T_1 are transverse along the evaluation maps at the endpoints, we obtain by Lemma 4.1 that T^2 is represented by the cycle

$$\begin{aligned}\Upsilon' := \{(\gamma,\delta) \in Y_1 \times Y_1 \,:\, &\gamma(0) \in \mathbb{P}_1,\ \gamma(1) = \delta(0) \in h_{\gamma(0)}^{\mathbb{P}_1},\ \delta(1) \in h_{\delta(0)}^{\mathbb{P}'_1},\\ &\mathrm{im}(\gamma) \subset \ell^{+}_{\gamma(0)} \text{ and } \mathrm{im}(\delta) \subset \ell'^{+}_{\delta(0)}\}.\end{aligned}$$

Note that $\mathbb{P}_1 \cap \mathbb{P}'_1 = \mathbb{P}(\mathbb{R}^{2t-2} \times \{0\} \times \mathbb{R} \times \{0\}) =: \mathbb{P}_2$. Backwards interpolation from $\mathbb{P}'_1$ to $\mathbb{P}_1$ provides the homologous cycle

$$\begin{aligned}\Upsilon := \{(\gamma,\delta) \in Y_1 \times Y_1 \,:\, &\gamma(0) \in \mathbb{P}_1,\ \gamma(1) = \delta(0) \in h_{\gamma(0)}^{\mathbb{P}_1},\ \delta(1) \in h_{\gamma(0)}^{\mathbb{P}_1},\ \gamma(1) \in \mathbb{P}_2,\\ &\mathrm{im}(\gamma),\ \mathrm{im}(\delta) \subset \ell^{+}_{\gamma(0)}\}.\end{aligned}$$

Yet another description of the cycle Υ that represents T^2 is the following: it consists of triples $(x_0, x_1, x_2) \in \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ such that x_0, x_1, x_2 belong to the same Hopf fiber $h_{x_0}^{\mathbb{P}_1}$ and $x_1 \in \mathbb{P}_2$. Indeed, such a triple determines uniquely a concatenated path from the vertical half-circles connecting x_0 to x_1 and x_1 to x_2 inside $\ell^{+}_{x_0}$. This description also shows that Υ is a manifold: the space of triples (x_0, x_1, x_2) which belong to the same Hopf fiber is the double pull-back of the projectivized Hopf fibration via the projection map. Imposing that x_1 belongs to $\mathbb{P}_2$ amounts to taking the preimage of $\mathbb{P}_2$ via the second projection, which is a submersion.

Proof that $T^2 = T$.

The cycle Υ represents T^2 via the concatenation $C_{x_0,x_1} \cdot C_{x_1,x_2}$ of the two vertical half-circles C_{x_0,x_1} and C_{x_1,x_2} that connect x_0 to x_1, respectively x_1 to x_2 inside $\ell^{+}_{x_0}$. This concatenation can be homotoped to C_{x_0,x_2} inside $\ell^{+}_{x_0}$ by concatenating the unique arc of vertical half-circle running from x_0 to $C_{x_1,x_2}(t)$ and the arc of half-circle $C_{x_1,x_2}|_{[t,1]}$, for $t \in [0,1]$. (See Figure 6. Note that this procedure is reminiscent of the proof of Lemma 4.12.)

Thus T^2 is represented by the map

$$C_{x_0,x_2} : \Upsilon \to \mathcal{P}_n, \qquad (x_0, x_1, x_2) \longmapsto C_{x_0,x_2}.$$

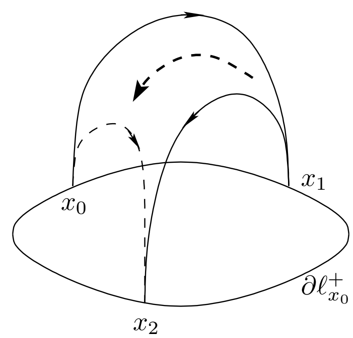


FIGURE 6. Deforming Υ to T_1.

Adopting the description (24) for the cycle T_1 that represents the class T, we obtain a factorization

$$\begin{tikzcd} \Upsilon \arrow[r, "C_{x_0,x_2}"] \arrow[d, "p_{02}"'] & \mathcal{P}_n \\ T_1 \arrow[ur, "C_{x_0,x_2}"'] & \end{tikzcd} \tag{25}$$

with $p_{02} : \Upsilon \to T_1$, $(x_0, x_1, x_2) \longmapsto (x_0, x_2)$. The key point is that the map p_{02} is smooth and generically one-to-one. Indeed, the subspace $\mathbb{P}_2$ (on which the point x_1 is constrained to lie) contains entirely the Hopf fibers which lie in $\mathbb{P}(\mathbb{R}^{2t-2} \times \{0\})$, but intersects in exactly *one* point the Hopf fibers which are not contained in $\mathbb{P}(\mathbb{R}^{2t-2} \times \{0\})$.

Since the horizontal arrow in (25) represents T^2 and the diagonal arrow represents T, we infer that $T^2 = T$. $\square$

This proves Theorem 1.1 in the case n even, which was the last remaining case.

Remark 4.6. The reader is invited to compare the proofs of the identities $S^2 = 0$ and $T^2 = T$, respectively $HS + SH = U$ and $TH + HT = H$ in view of the discussion on heredity in §4.1(4).

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