Surgery and tightness in contact 3-manifolds

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ABSTRACT. This article is an expository overview of the proof of the author that tightness of a closed contact 3-manifold is preserved under Legendrian surgery. The aim is to give a somewhat more leisurely, motivated, and illustrated version of the ideas and constructions involved in the original paper.

1. Introduction

The general context of this article is the story of porting the tools and methods of low-dimensional topology to a more modern setting, in particular the world of complex, almost complex, or symplectic 4-manifolds, with contact 3-manifold boundaries (the interested reader may consult [CE] for an exhaustive overview of these structures and the interplay between them). Following the development of Donaldson theory [D], and related discoveries such as the Seiberg-Witten equations and Taubes's Gromov invariants [T], such structures have a prominent role to play in the study of smooth 4-manifolds; one is thus interested in being able to apply the 'cut and paste' techniques of the smooth category in the geometric setting. We think of cutting a 4-manifold into pieces, each a cobordism between 3-manifolds, in a way which respects the geometry. Doing so then induces a contact structure on each boundary component, so the 'pasting' is an identification of contact 3-manifolds, the role of which is in some sense to keep track of the geometric information.

For the purposes of this paper, we concern ourselves with the attachment of a 4-dimensional 2-handle. This operation plays a central role in the classical study of low-dimensional smooth manifolds; a 4-dimensional 2-handlebody closes uniquely (by Laudenbach and Poenaru [LP]), while of course each smooth 4-manifold admits such a description. For various reasons, the 0- and 1- handles are not particularly interesting, and one is left to understand the 2-handles. We thus focus on an elementary cobordism gotten by thickening up a 3-manifold and adding a 2-handle, the trace of which on the boundary manifold is a surgery on a knot. Much of the set-up of low-dimensional topology, e.g. intersection forms of 4-manifolds, the Kirby calculus, Rolfsen surgery, link invariants of 3-manifolds, and so on, are all based on this operation.

In the geometric setting, a very similar story has taken shape. Again the 0- and 1-handles have little role to play, so we focus on the 2-handles. It was shown by Weinstein [We] (for the symplectic case) and Eliashberg [El2] (for the Stein case) that by attaching a 2-handle to a thickened *contact* 3-manifold in a particular way (to be explained further in the paper) the resulting cobordism then carries the desired geometry, and so induces a contact structure on the new 'convex' end. The resulting surgery on the boundary is then referred to a *Legendrian* surgery, and plays much the role in the contact category that Dehn surgery plays in the smooth category.

The story thus seems to translate as well as one may hope; a potential problem though arises in application. As it happens, contact structures come in two varieties: *tight* and *overtwisted*, the overtwisted ones being those for which (by Eliashberg [El1]) an 'h-principle' holds, which in particular implies that they are (up to isotopy) completely determined by their homotopy classes as plane fields. As such, we think of overtwisted structures as the 'non-geometric' ones, as they are incapable of carrying any 'modern' information.

One would then like to know if, building a Stein cobordism from a tight structure, the resulting convex end is again tight. This question was answered (in the affirmative) by our paper [W2]; the purpose of this article is to give a somewhat more motivated (and illustrated) overview of some of the ideas and constructions involved.

The main tool utilized is the correspondence theorem of Giroux, which establishes a 1-1 correspondence between isotopy classes of contact structures, and stabilization classes of open book decompositions. We give a characterization of the tightness condition in this setting, showing:

Theorem 4.1. *Let M be a closed 3-manifold, ξ a contact structure. Then the following are equivalent:*

(1) *ξ is tight.*
(2) *Some open book decomposition supporting (M, ξ) is consistent.*
(3) *Each open book decomposition supporting (M, ξ) is consistent.*

Our main result is then to use this characterization to demonstrate:

Theorem 5.1. *If (M, ξ) is obtained by Legendrian surgery on tight (M', ξ'), for M' a closed 3-manifold, then (M, ξ) is tight.*

2. Preliminaries

Let M denote a closed, oriented three-manifold, and ξ a positive co-oriented contact structure on M; i.e. ξ is the kernel of some global 1-form α satisfying the condition that $\alpha \wedge d\alpha$ is a positive volume form on M. Put yet another way, ξ is a positive, co-oriented, nowhere integrable plane field. The structure is *overtwisted* is there is some embedded disc $D \hookrightarrow M$ such that, at each point $p \in \partial D$, we have $T_p D = \xi_p$. Otherwise ξ is *tight*.

An *open book decomposition* of M is a pair (B, π) consisting of an embedded oriented link $B \hookrightarrow M$, and fibration $\pi : M \setminus B \to S^1$ such that each fiber is the interior of an
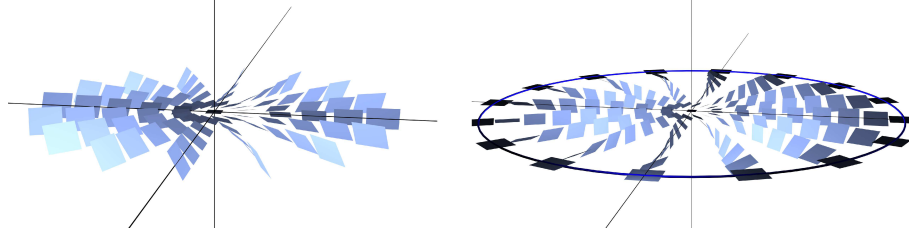
FIGURE 1. To the left, the 'standard' tight contact structure on $\mathbb{R}^3$, with cylindrical coordinates (r, θ, z), defined by $dz + r^2 d\theta = 0$. To the right, the standard *overtwisted* structure $\cos r dz + r \sin r d\theta = 0$; the blue circle bounds an obvious overtwisted disc.

oriented surface with boundary B (i.e. a *Seifert surface* for B). For our purposes we will often find it more useful to keep track only of the abstract data associated with this pair, that is the Seifert surface Σ, and the monodromy map $\varphi \in \pi_0 Diff^+(\Sigma, \partial)$ of the fibration; M is recovered from this data by forming the mapping torus $\Sigma_\varphi := \Sigma \times [0,1] \big/ _{(p,1) \sim (\varphi(p), 0)}$, and then filling in the boundary $\partial\Sigma \times S^1$ with $\partial\Sigma \times D^2$ in the obvious way. The surface Σ is referred to as the *page* of the decomposition, while B is the *binding*.
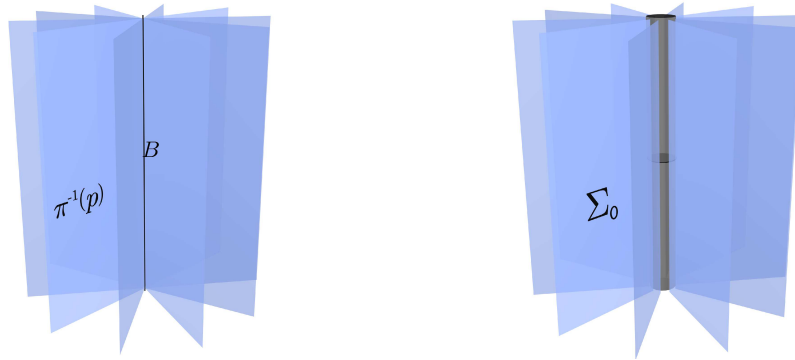

FIGURE 2. The two points of view of an open book decomposition. To the left, B is a fibered link in M, while to the right we reconstruct M from the abstract data (Σ, φ).

Of particular importance to us is the operation of *stabilization* of an open book, gotten by plumbing a (positive/negative) Hopf band, as follows:

Definition 2.1. Let (Σ, φ) be an open book decomposition of M, and σ a properly embedded arc in Σ. Let Σ' denote the surface given by attaching a 1-handle to Σ with

attaching sphere $\partial\sigma$, and denote by $s \subset \Sigma'$ the simple closed curve gotten by taking the union along the boundary of σ with the core of the new handle. Then the pair $(\Sigma', \tau_s^\pm \circ \varphi)$, where $\tau_s^\pm$ denotes the positive/negative Dehn twist about s, and φ the obvious inclusion of the original φ (extended over the handle by the identity), is again an open book decomposition of M, referred to as a positive/negative *stabilization* of (Σ, φ), via σ.
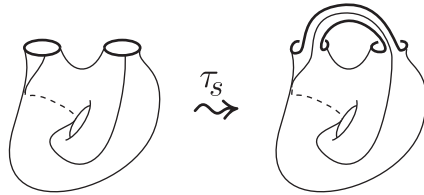

FIGURE 3. Stabilizing (Σ, φ) via σ results in $(\Sigma \cup h, \tau_S \circ \varphi)$.

Open book decompositions were introduced by Alexander in [A], where it was shown that every 3-manifold admits such a decomposition. As it later turned out, they are particularly well-adapted to keeping track of contact structures. Indeed, consider the plane field on M which is tangent to the pages, and 'flips' passing across a meridional disc of the binding (so, e.g. again using cylindrical coordinates for one of the solid tori, with dz a volume form for the binding and (r, θ) coordinates on the unit-disc cross-section, we may define our plane field in the torus by $\ker((1-r)dz + r^2 d\theta)$.
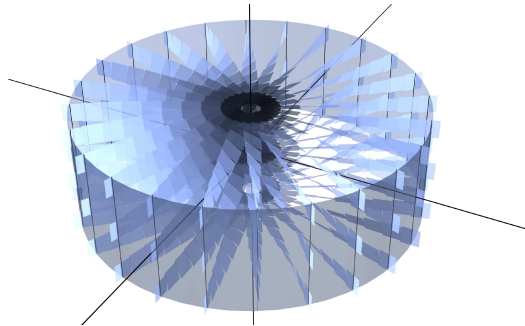

FIGURE 4. The plane field associated to an open book, in a slice of the thickened binding.

Thurston and Winkelnkemper [TW] demonstrated that the homotopy class of this plane field contains a contact structure (intuitively, one 'relaxes' the twisting, allowing it to continue across the page), thus providing a short proof that all 3-manifolds admit

contact structures. This relation was then studied systematically by Giroux, culminating in the following definition and theorem:

Definition 2.2. [Gi] A contact structure $\xi = \ker \alpha$ is *supported by* open book (B, π) if α is a positive volume form for B, and $d\alpha$ a positive volume form for each fiber $\pi^{-1}(p)$.

Theorem 2.1. [Gi] *Let M be a closed, oriented, smooth 3-manifold. Then there is a 1-1 correspondence between contact structures on M up to isotopy, and open book decompositions of M supporting ξ up to isotopy and positive stabilization.*

3. Overtwisted discs in open book decompositions

3.1. Some historical background

Given the above then, it is natural to wonder if tightness/overtwistedness of a given contact structure can be deduced from the data of a supporting open book decomposition. There are historically two directions of approach to the problem, which we can distinguish as on the one hand using global properties to obstruct existence of overtwisted discs (i.e. detect tightness), or on the other using local information to exhibit such discs (i.e. detect overtwistedness). For motivational purposes we will briefly sketch these approaches:

The first approach, toward detecting tightness, falls out of work of (in rough order of chronological contribution) Gromov, Eliashberg, and Giroux. The rough idea is to show that, given open book decomposition (Σ, φ) supporting (M, ξ), if φ admits a factorization into positive Dehn twists, then one may use this to build a Lefschetz fibration of a Stein domain X whose boundary is M. It then follows from Loi and Piergalinni [LP], Akbulut and Ozbagci [AO], and Plamenevskaya [P], that ξ is isotopic to the contact structure on M given by the complex tangencies of X. It follows then (from work of Gromov [Gr] and Eliashberg [El3]) that ξ is tight. We summarize as follows:

$$\begin{array}{l} \varphi \text{ admits a factorization} \\ \text{into positive Dehn twists} \end{array} \Rightarrow (M, \xi) \text{ is Stein fillable} \Rightarrow \xi \text{ is tight}$$

Unfortunately though, the converse holds for neither of these implications. Indeed, Eliashberg first produced examples of tight structures which are not Stein fillable, while in [W1] we exhibited examples of open book decompositions supporting Stein fillable structures but such that the monodromy admits no positive factorization.

The second approach, that of detecting an overtwisted disc, again starts from the above mentioned theorems of Eliashberg and Giroux, and follows the general method of using properly embedded arcs in surfaces to generate mapping class invariants. Before going any further, we lay out some conventions concerning such arcs, and their representations in figures. To begin, throughout the remainder of the article, an *arc* will refer to a properly embedded arc in the page of an open book decomposition, and will be represented by a straight line in figures. The image of an arc under the monodromy map will be represented by a curved line, and if the arc is oriented, its image will be given the opposite orientation. Thus, given a collection of oriented arcs Γ in open book decomposition (Σ, φ), we may

assign a sign to each point $p \in \Gamma \cap \varphi(\Gamma)$ by considering the ordered pair of vectors at p along Γ and $\varphi(\Gamma)$; if this pair gives the standard orientation of Σ at p, we say p is *positive*, otherwise p is negative.
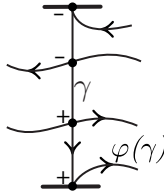

FIGURE 5

We start then with the observation that, given Eliashberg's classification, and the fact (due to Neuman and Rudolph [NR]) that a negative stabilization acts on the homotopy class of the associated plane field by a $+1$ 'shift' of the 'enhanced Milnor number', it follows immediately that any overtwisted structure is supported by an open book decomposition that is a negative stabilization of some other. From the point of view of one actually trying to detect this condition, we observe that we may characterize it as follows:

Definition 3.1. Let (Σ, φ) be an open book decomposition, and γ a properly embedded arc in Σ. We say γ is *negative-destabilizable* if

(1) $\gamma \cap \varphi(\gamma) \cap \partial\Sigma$ is a pair of negative points, and
(2) $\gamma \cap \varphi(\gamma) \cap int(\Sigma) = \emptyset$.

Existence of such an arc is clearly equivalent to (Σ, φ) being a negative stabilization, with γ the co-core of the stabilization 1-handle. As the property of being a negative stabilization easily implies connect sum with an overtwisted structure on S^3, we then have the following equivalence:

Corollary 3.1. *A contact structure is overtwisted if and only if there exists some supporting open book decomposition (Σ, φ) with a negative-destabilizeable arc.*

This observation was then generalized by Goodman [Go], who makes the following definition:

Definition 3.2. Let (Σ, φ) be an open book decomposition, and γ a properly embedded arc in Σ. We say γ is *sobering* if

(1) at least one point of $\gamma \cap \varphi(\gamma) \cap \partial\Sigma$ is negative, and
(2) each $p \in \gamma \cap \varphi(\gamma) \cap int(\Sigma)$ is negative.

Now, Definition 3.2 clearly generalizes Definition 3.1, so any overtwisted contact structure is supported by an open book with a sobering arc. It turns out that the condition is sufficient, as well, as Goodman shows by suspending such an arc in the open book decomposition (see the proof of Theorem 3.4), and extending this to a surface which violates

the Thurston-Bennequin inequalities, which generalize overtwisted discs and in particular are known to be satisfied in tight manifolds. We arrive at:

Theorem 3.2. *A contact structure is overtwisted if and only if there exists some supporting open book decomposition (Σ, φ) with a sobering arc.*

A final generalization was then given by Honda-Kazez-Matic [HKM1], showing that, considering minimally intersecting pairs $\gamma \cap \varphi(\gamma)$, Goodman's second condition is actually unnecessary, as follows:

Definition 3.3. Let (Σ, φ) be an open book decomposition, γ a properly embedded arc in Σ, and p an endpoint of γ. We say γ is *mapped to the left (at p)* if, after an isotopy of $\varphi(\gamma)$ minimizing $\gamma \cap \varphi(\gamma)$, p is negative.

Again, that any overtwisted contact structure is supported by an open book with such an arc is immediate; the content of [HKM1] is thus to show, using the 'bypass' theory' developed previously by Honda, that the condition is sufficient, giving:

Theorem 3.3. *A contact structure is overtwisted if and only if there exists some supporting open book decomposition with an arc which is mapped to the left at an endpoint.*
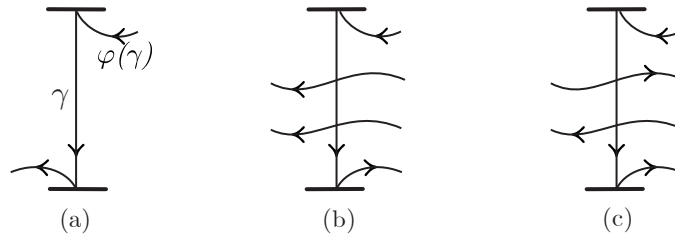

FIGURE 6. Examples of arcs which are (a) a negative stabilization, (b) sobering, and (c) mapped to the left at an endpoint.

(A mapping class which maps no arc to the left is referred to as *right veering*. We refer the reader to our paper [W4] for a simple algorithm to stabilize any non-right-veering open book decomposition into one which contains a particularly simple sobering arc, thus giving an elementary alternate proof of Theorem 3.3, and a substantial simplification of the proof of Theorem 3.2).

Again however none of these criteria completely characterizes overtwistedness in terms of an *arbitrary* open book decomposition. Indeed, it is straightforward to see that any open book decomposition can be stabilized to be right-veering.

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3.2. Overtwisted regions

The point to the remainder of this section then is to introduce a somewhat different (almost) generalization of Definition 3.1, and to explain explicitly how it implies overtwistedness.

Definition 3.4. Let (Σ, φ) be a open book decomposition, and Γ a collection of disjoint, oriented arcs in Σ such that each point of $\partial\Gamma$ is positive in $\Gamma \cap \varphi(\Gamma)$. An *overtwisted region* (in $(\Sigma, \varphi, \Gamma)$) is an embedded disc $A \hookrightarrow \Sigma$, with $\partial A \hookrightarrow (\Gamma \cup \varphi(\Gamma))$, such that:

(1) Corners of A alternate between points in $\partial\Gamma$, and negative points in the interior of Σ.
(2) Each point of $\Gamma \cap \varphi(\Gamma) \cap int(\Sigma)$ is a corner of A.
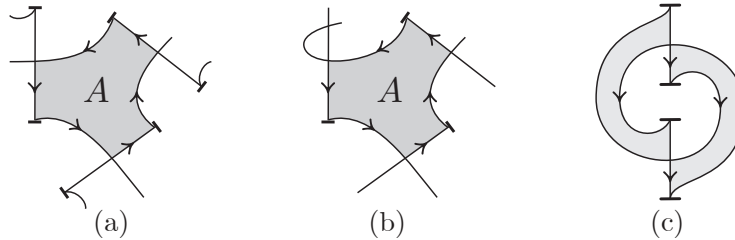(3) A is the unique such disc.

FIGURE 7. (a) An overtwisted region. (b) A disc satisfying (1) but not (2). (c) Each of the illustrated discs satisfy (1) and (2), but not (3).

Remark 3.1. It should be emphasized that there is no assumption concerning minimality of $\Gamma \cap \varphi(\Gamma)$; in particular, for the case that Γ contains only a single arc, an overtwisted region is a bigon.

As a justification for the terminology, we have the following lemma:

Lemma 3.4. *Suppose* $(\Sigma, \varphi, \Gamma)$ *has an overtwisted region. Then the supported contact structure is overtwisted.*

Proof. We begin by recalling that an open book decomposition decomposes M into two parts: the mapping torus $\Sigma_\varphi = (\Sigma \times [0,1])/_{(p,1)\sim(\varphi(p),0)}$, and a collection of solid tori filling in the boundary. Motivated by Goodman's proof, we consider then the restriction D of this construction to an element $\gamma \in \Gamma$; that is

$$D := (\gamma \times [0,1])/\sim \bigcup \{2 \text{ meridional discs}\}$$

In words, D is constructed from a rectangle $(\gamma \times [0,1])$, the corners of which are then identified in two pairs by $\sim$, resulting in an object which has the homotopy type of a

sphere with three holes, two of which are capped by meridional discs of the solid tori. The remaining boundary component is the simple closed curve on $\Sigma_0 := \Sigma \times \{0\}$ given by $\gamma \cup \varphi(\gamma)$, which we smooth and push into the interior of the page.
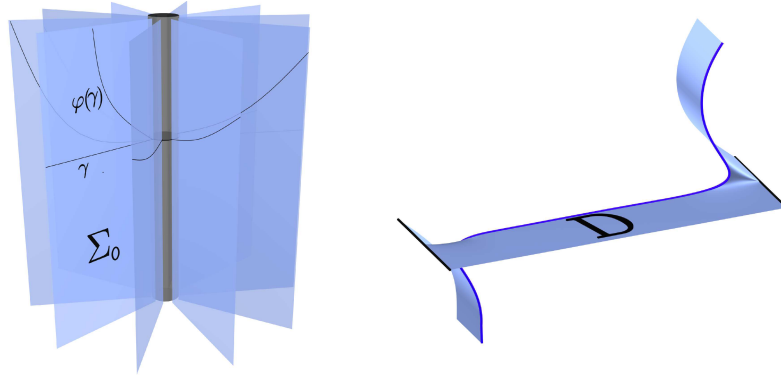

FIGURE 8. To the left, the suspension of γ in the open book decomposition, in a neighborhood of an endpoint of γ. To the right, the suspension by itself, in a neighborhood of γ. The blue boundary component is thus $\gamma \cup \varphi(\gamma)$, pushed to lie in the interior of Σ_0, while each of the two black components closes to bound a meridional disc of the binding over which D extends.

Repeating the construction on each element of Γ, we have a collection of embedded discs $\{D_i\}$ in M, each with boundary on $\Sigma \times \{0\}$, and such that each intersects the next in exactly one point (a negative corner of A). Again following Goodman, we observe that each such intersection may be resolved by a surgery, the topological realization of which may be seen as pushing the discs slightly apart and connecting by a twisting ribbon for each such point, so as to preserve the orientations (Figure 9). This resolution of $\cup_i D_i$ is then an embedded cylinder in M, one boundary component of which bounds the overtwisted region A, the other a homotopically non-trivial curve on Σ_0.

As such, capping the cylinder with A, we obtain an embedded disc with boundary on the page, whose disc framing matches the framing it gets from the page. It follows then after using the standard Legendrian realization techniques of Giroux and Honda that our disc is in fact an overtwisted disc in (M, ξ).

$\square$

As it turns out, existence of an overtwisted region not only implies overtwistedness of the supported contact structure, but also 'stably' completely characterizes the condition, as follows:
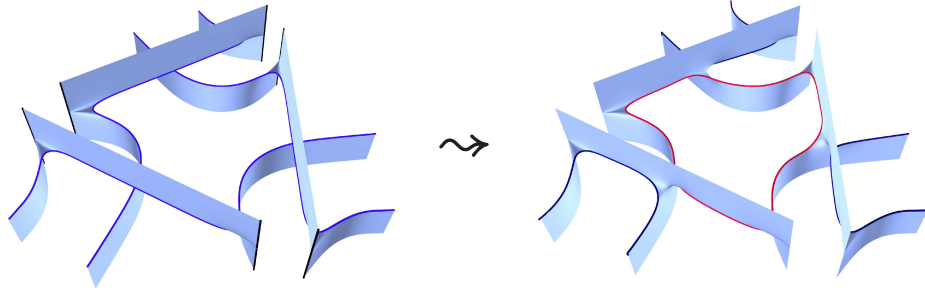
FIGURE 9. Resolving $\cup_i D_i$.

Definition 3.5. A class $\varphi \in MCG(\Sigma)$ is *inconsistent* if there is some arc collection Γ in (Σ, φ) such that, stably, $(\Sigma, \varphi, \Gamma)$ has an overtwisted region. Otherwise, φ is *consistent*.

Here a stabilization of a triple $(\Sigma, \varphi, \Gamma)$ is the triple $(\Sigma \cup h, \tau_s \circ \varphi, \iota(\Gamma))$, where $(\Sigma \cup h, \tau_s \circ \varphi)$ is the usual data associated to a stabilization, and ι the obvious inclusion induced by any stabilization done so that the attaching sphere of the handle is disjoint from $\partial\Gamma$. Thus 'stably' simply means 'there exists a sequence of positive stabilizations such that'.

4. Characterizing tightness

We turn now to the proof of Theorem 4.1, whose statement we recall:

Theorem 4.1. *Let M be a closed 3-manifold, ξ a contact structure. Then the following are equivalent:*

(1) *ξ is overtwisted.*
(2) *Some open book decomposition supporting (M, ξ) is inconsistent.*
(3) *Each open book decomposition supporting (M, ξ) is inconsistent.*

In fact, we will find it convenient to add a couple of further equivalent statements to the theorem; in particular:

(4) For any open book decomposition (Σ, φ) supporting ξ, and any basis $\mathcal{B}$ and curve system L in Σ, $\mathcal{B}$ stably detects overtwistedness relative to L.
(5) There exists an open book decomposition (Σ, φ) supporting ξ, and basis $\mathcal{B}$ of Σ, such that for any curve system L in Σ, $\mathcal{B}$ stably detects overtwistedness relative to L.

As for the new vocabulary contained in these new statements, a *basis* of a surface with boundary is a collection of simply embedded arcs which cut the surface into a disc, while a *curve system* is any collection of properly embedded curves and arcs which are disjoint in the interior of the surface. Finally, we say a basis $\mathcal{B}$ *stably detects overtwistedness relative to* a curve system L if there is an arc collection Γ, sequence S of stabilizations (here S

actually denotes the composition of the Dehn twists associated to the stabilizations), and subsequence S' of S, such that:

- Each element of Γ is isotopic to one of $\mathcal{B}$,
- $(\Sigma \cup H, S \circ \varphi, \Gamma)$ has an overtwisted region A, and
- there is some negative corner y of A such that , for any neighborhood U_y of y, $S'(L)$ can be isotoped such that $S'(L) \cap (\Gamma \cup S(\varphi(\Gamma))) \subset U_y$.

We pause in an attempt at giving some intuition to the theorem. In short, the thing to take away is as follows: Suppose we know that a given contact structure is overtwisted, and we have an arbitrary supporting open book decomposition, some basis of the surface, and some curve system. The theorem then says that we can always find a collection of arcs, each isotopic to some element of the basis, and then stabilize the open book such that, in the stabilization, the arcs and their images determine an overtwisted region. Moreover, there is some subsequence of the Dehn twists associated to the stabilizations such that the image of our curve system under the subsequence intersects the arcs and their images in a particularly simple way, in particular *all* such intersections occur in a neighborhood of a single corner of the region.
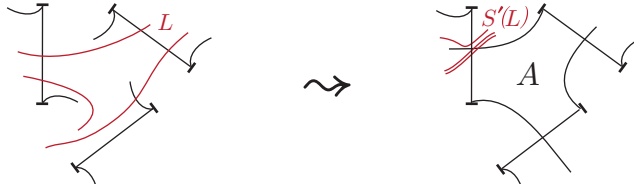

FIGURE 10. Some arbitrary initial open book (Σ, φ), basis B and curve system L, is stabilized into $(\Sigma \cup H, S \circ \varphi, \Gamma)$ satisfying all desired properties.

Sketch of proof. The implications $(4) \Rightarrow (3) \Rightarrow (2)$ are immediate, while $(2) \Rightarrow (1)$ follows from Lemma 3.4.

$(1) \Rightarrow (5)$ Using the characterization of overtwistedness in terms of negative stabilizations (Corollary 3.1), we observe that if an arc γ is negatively-destabilizable in some open book decompositon, $\mathcal{B}$ any basis containing γ, and L any curve system, then after the sequence of three stabilizations illustrated in Figure 11 we achieve all conditions of statement (5) (note that by attaching the handles in a sufficiently small neighborhood of $\partial\gamma$ we ensure that any points of $L \cap \partial\Sigma$ are away from the picture, and thus irrelevant).

$(5) \Rightarrow (4)$ There are two parts to prove: That the property of detecting overtwistedness relative to a given curve system in a given open book decomposition is independent of the choice of basis, and then that it is moreover independent of the choice of open book decomposition. The proof of the first comes from considering 'arc-slides' in the surface. It is known that any two bases are related by such slides, so it suffices to show that these
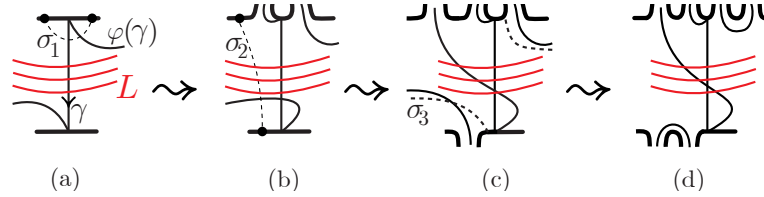
FIGURE 11. The stabilization arcs are (a) boundary parallel, (b) parallel (in the original surface Σ) to γ, and (c) parallel to $\varphi(\gamma)$.

preserve all desired properties. We refer to the original paper [W2] for details. Once we have this, though, the second statement becomes nearly trivial. In particular, as Giroux's theorem (Theorem 2.1) tells us that all supporting open book decompositions are related by positive stabilization and destabilization, it is left only to show that these operations preserve our properties. However, as we are free to work with any basis, we may choose one whose image is unaffected by a given stabilization or destabilization. As stabilization and destabilization each send curve systems to curve systems (note in particular that this would not follow if we considered only closed curves), the result is immediate.

$\square$

5. Legendrian surgery

With the statement of Theorem 4.1 fresh in our minds, we turn to the proof of Theorem 5.1:

Theorem 5.1. *If (M, ξ) is obtained by Legendrian surgery on tight (M', ξ'), for M' a closed 3-manifold, then (M, ξ) is tight.*

We briefly recall the set-up: given (M', ξ'), and Legendrian knot L in M' (recall that this simply means that the tangent bundle of L is contained in ξ), we start with the trivial cobordism $M' \times [0, 1]$, then add a 2-handle to $M' \times \{1\}$, with attaching sphere $L \times \{1\}$, and framing (-1) relative to ξ_L. The result is then a Stein cobordism with concave end (M', ξ'), and convex end (M, ξ).

We begin by translating the problem into the language of open book decompositions. We require:

Lemma 5.2. *(Giroux - see e.g. [Et], Corollary 4.23) There exists an open book decomposition (Σ, φ) supporting (M, ξ) such that L is a homologically non-trivial curve in a page (in fact, for our purposes, we only require homotopical non-triviality).*

Using this, it then follows easily from standard low-dimensional topological methods (see e.g. [L]) that the open book decomposition $(\Sigma, \tau_L^{-1} \circ \varphi)$ supports (M', ξ'). Bringing all of this together, we re-write Theorem 5.1 as follows:

Theorem 5.3. *Let $\varphi \in MCG(\Sigma)$ be inconsistent, and $L \subset \Sigma$ a homotopically non-trivial simple closed curve. Then $\tau_L^{-1} \circ \varphi$ is again inconsistent.*

Proof. Following Theorem 4.1, we may find an arc collection Γ, and stabilizations $S = \tau_{s_n} \cdots \tau_{s_2} \tau_{s_1}$, such that the stabilized triple $(\Sigma \cup H, S \circ \varphi, \Gamma)$ has overtwisted region A. Moreover, we can find some subsequence S' of S, and a negative corner y of A, such that $S'(L) \cap (\Gamma \cup S(\varphi(\Gamma)))$ is contained in an arbitrary neighborhood of y. On the other hand, we know from the above fact that $(\Sigma, \tau_L^{-1} \circ \varphi)$ is an open book decomposition supporting (M', ξ').

We consider then a somewhat more general set-up: supposing that one starts with an open book decomposition (Σ, φ), and some arc σ, and simple closed curve L in the page. One may then use these to modify the open book data in two ways: stabilization via σ, or composition of the monodromy with a negative Dehn twist about L, which we will refer to as *surgering* the open book *via L*. Now, using the well-known (and easily verifiable) fact that the map one obtains by conjugating a Dehn twist about a curve by a surface diffeomophism is isotopic to the Dehn twist about the image of the curve under the diffeomorphism, one can easily verify that the open book one gets by stabilizing first via σ and then surgering via L is in fact the same open book that one gets by first surgering via L, and then stabilizing via $\tau_L^{-1}\sigma$:

$$
\begin{array}{ccc}
(\Sigma, \varphi) & \xrightarrow{\text{surgery via } L} & (\Sigma, \tau_L^{-1} \circ \varphi) \\
\downarrow {\scriptstyle \text{stabilization via } \sigma} & & \searrow {\scriptstyle \text{stabilization via } \tau_L^{-1}(\sigma)} \\
(\Sigma \cup h, \tau_s \circ \varphi) & \xrightarrow{\text{surgery via } L} & (\Sigma \cup h, \tau_L^{-1} \circ \tau_s \circ \varphi) = (\Sigma \cup h, \tau_{\tau_L^{-1}(s)} \circ \tau_L^{-1} \circ \varphi)
\end{array}
$$

Similarly, first surgering via L and secondly stabilizing via σ is equivalent to first stabilizing via σ and secondly surgering via $\tau_s(L)$ (where, as usual, s refers to the closed curve gotten by extending σ over the stabilization handle):

$$
\begin{array}{ccc}
(\Sigma, \varphi) & \xrightarrow{\text{surgery via } L} & (\Sigma, \tau_L^{-1} \circ \varphi) \\
\downarrow {\scriptstyle \text{stabilization via } \sigma} & & \searrow {\scriptstyle \text{stabilization via } \sigma} \\
(\Sigma \cup h, \tau_s \circ \varphi) & \xrightarrow{\text{surgery via } \tau_s(L)} & (\Sigma \cup h, \tau_{\tau_s(L)}^{-1} \circ \tau_s \circ \varphi) = (\Sigma \cup h, \tau_s \circ \tau_L^{-1} \circ \varphi)
\end{array}
$$

Using this observation, then, it is straightforward in our particular setup to find a stabilization sequence $\widehat{S}$ such that stabilizing $(\Sigma, \tau_L^{-1} \circ \varphi)$ (which, recall, supports (M', ξ')) with the sequence $\widehat{S}$ is equivalent to surgering $(\Sigma \cup H, S \circ \varphi)$ (which supports (M, ξ), and has overtwisted region A) via $\tau_{S'(L)}$ (Figure 12).

We are left then to consider the augmented open book decomposition given by $(\Sigma \cup H, \tau_{S'(L)}^{-1} \circ S \circ \varphi, \Gamma)$. It is immediate (again referring to Figure 12) that this triple determines a region (which we again refer to as A) which satisfies all conditions of an
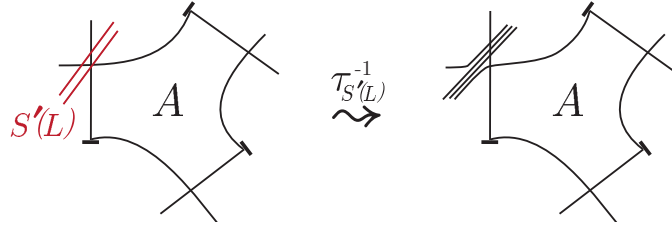
FIGURE 12. The effect of surgery via $S'(L)$ on the overtwisted region A in $(\Sigma \cup H, S \circ \varphi)$, giving the configuration to the right in an open book supporting (M', ξ').

overtwisted region other than the condition on intersections of arcs and their images in the interior of the page (condition (2) of Definition 3.4). As such, the result will follow if we are able to remove all such intersections. Indeed, an algorithm to do just this can be found in [W4]. We will however follow our original argument from [W2], which while being somewhat less self-reliant is also substantially quicker.

Consider then firstly the case that A is a bigon - clearly then $\tau_{S'(L)}^{-1} \circ S \circ \varphi$ maps the arc γ to the left (Definition 3.3), and so the result follows directly from that of Honda, Kazez, and Matic (Theorem 3.3).

It turns out though that one may in fact always reduce to this case. To see this, let γ denote the element of Γ containing the point y, and consider some other element γ' of Γ. It is then clear that γ' is the co-core of a stabilization 1-handle, and so may be destabilized. Focusing then on a neighborhood of γ' (Figure 13), let s denote the simple closed curve obtained by pushing $\gamma' \cup S(\varphi(\gamma'))$ into the interior of Σ. The destabilization is then obtained by composing φ with a negative Dehn twist about s (Figure 13(b)), and cutting Σ along γ'. As $s \cap S(\varphi(\Gamma))$ is a single point, all of this can be kept track of in this neighborhood; we leave it to the reader to observe that the effect is to reduce the number of edges of A by 2, as indicated in the figure. Iterating the process, we arrive at a bigon, completing the proof.
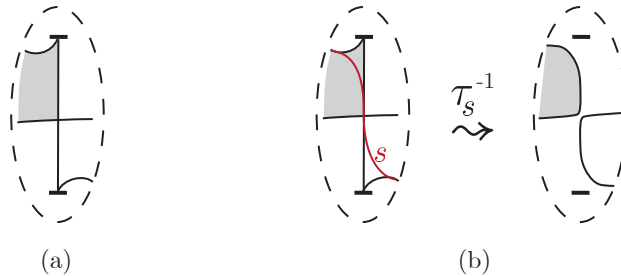

(a)   (b)

FIGURE 13

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□

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