

Noncommutative augmentation categories

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ABSTRACT. To a differential graded algebra with coefficients in a noncommutative algebra, by dualisation we associate an A_∞ -category whose objects are augmentations. This generalises the augmentation category of Bourgeois and Chantraine [2] to the noncommutative world.

1. Introduction

Differential graded algebras (DGAs for short) were introduced by Cartan in [4] and occur naturally in a number of different areas of geometry and topology. We are here interested in those that appear in the context of Legendrian contact homology, which is a powerful contact topological invariant due to Chekanov [6] and Eliashberg, Givental and Hofer [18]. In its basic setup, this theory associates a differential graded algebra, called the *Chekanov-Eliashberg DGA*, to a given Legendrian submanifold of a contact manifold. The DGA homotopy type (or even, stable tame isomorphism type) of the Chekanov-Eliashberg DGA is independent of the choices made in the construction and invariant under isotopy through Legendrian submanifolds. Because of some serious analytical difficulties, Legendrian contact homology has been rigorously defined only for Legendrian submanifolds of contactisations of Liouville manifolds [15] and in few other sporadic cases [6, 21, 32, 26, 17].

Since the Chekanov-Eliashberg DGA is semifree and fully noncommutative, it can be difficult to extract invariants from it. In fact, as an algebra, it is isomorphic to a tensor algebra (and therefore is typically of infinite rank) and its differential is nonlinear with respect to the generators.

To circumvent these difficulties, Chekanov introduced his linearisation procedure in [6]: to a differential graded algebra equipped with an augmentation he associates a chain complex which is generated, *as a module*, by the generators of the DGA *as an algebra*. The differential then becomes linear at the price of losing the information which is contained in the multiplicative structure of the DGA, but at least the homology of the linearised

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complex is computable. It is well known that the set of isomorphism classes of linearised homologies is invariant under DGA homotopy; see e.g. [1, Theorem 2.8]. Thus, linearised Legendrian contact homology provides us with a computable Legendrian isotopy invariant.

In order to recover at least part of the nonlinear information lost in the linearisation, one can study products in the linearised Legendrian contact homology groups induced by the product structure of the Chekanov-Eliashberg DGA.

Civan, Koprowski, Etnyre, Sabloff and Walker in [8] endowed Chekanov’s linearised chain complex with an A_∞ -structure. This construction was generalised in [2] by the first author and Bourgeois, who showed that a differential graded algebra naturally produces an A_∞ -category whose objects are its augmentations. In dimension three, the A_∞ -category constructed by the first author and Bourgeois admits a unital refinement defined by Ng, Rutherford, Shende, Sivek and Zaslow in [29]. The latter article also establishes an equivalence between this unital A_∞ -category and one defined in terms of derived sheaves of microlocal rank one with microsupport given by a fixed Legendrian knot. Our expectation is that the A_∞ -structures constructed here correspond to such sheaves being of arbitrary microlocal rank.

A_∞ -algebras are by now classical structures which were first introduced by Stasheff in [36] as a tool in the study of ‘group-like’ topological spaces. Fukaya was the first to upgrade the notion of an A_∞ -algebra to that of an A_∞ -category. In [23] he associated an A_∞ -category, which now goes under the name of the *Fukaya category*, to a symplectic manifold. See [33] for a good introduction. Inspired by Fukaya’s work [23], Kontsevich in [25] formulated the *homological mirror symmetry conjecture* relating the derived Fukaya category of a symplectic manifold to the derived category of coherent sheaves on a “mirror” manifold.

The construction in [8] and [2] defines A_∞ -operations only when the coefficient ring of the DGA is commutative. The goal of this paper is to extend that construction to noncommutative coefficient rings in the following two cases:

- (I) the coefficients of the DGA as well as the augmentations are taken in a unital noncommutative algebra, or
- (II) the coefficients of the DGA as well as the augmentations are taken in a noncommutative *Hermitian algebra*. (See Definition 2.1.) This case includes both finite-dimensional algebras over a field and group rings.

Case (II) is obviously included in Case (I), but we will see that there is a particularly nice alternative construction of an A_∞ -structure in case (II) which gives a different result. We refer to Subsections 4.1 and 4.2 for the respective constructions. Both generalisations above are sensible to study when having Legendrian isotopy invariants in mind, albeit for different reasons.

Case (I) occurs because there are Legendrian submanifolds whose Chekanov-Eliashberg DGA does not admit augmentations in any unital algebra of finite rank over a commutative ring, but admits an augmentation in a unital noncommutative infinite-dimensional one (for example, in their characteristic algebras). The first such examples were Legendrian knots constructed by Sivek in [35] building on examples found by Shonkwiler and

Shea Vela-Vick in [34]. From them, the second and fourth authors constructed higher dimensional examples in [13]. Observe that any differential graded algebra has an augmentation in its “characteristic algebra”, introduced by Ng in [30], which is the quotient of the DGA by the two-sided ideal generated by its boundaries. This algebra is in general noncommutative and infinite-dimensional, and any augmentation factors through it. It is of course possible that the characteristic algebra vanishes, but it does so if and only if the DGA is acyclic [11]. The *complex* that we will define in case (I) (but not the higher order operations) was used in [12] by the second author in order to deduce that a Legendrian submanifold with a non-acyclic Chekanov-Eliashberg DGA does not admit a displaceable Lagrangian cap.

Finally, we note that the construction we give in Case (I) is closely related to the A_∞ -structures and bounding cochains with noncommutative coefficients as introduced by Cho, Hong and Sui-Cheong in their recent work (see [7]). Namely, the (uncurved) A_∞ -structures that we produce from a DGA and its augmentations can be seen to coincide with the (uncurved) A_∞ -structures produced by their bounding cochains.

Case (II) also occurs naturally in the context of Legendrian contact homology. For example, in [28] Ng and Rutherford show that augmentations of certain satellites of Legendrian knots induce augmentations in matrix algebras for the Chekanov-Eliashberg DGA of the underlying knot. Moreover, coefficients in a group ring appear naturally if one considers the Chekanov-Eliashberg DGA with coefficients “twisted” by the fundamental group of the Legendrian submanifold. We learned this construction from Eriksson-Östman, who makes use of it in his upcoming work [19]. This version of Legendrian contact homology can be seen as a natural generalisation of Morse homology and Floer homology with coefficients twisted by the fundamental group; see the work [37] by Sullivan and [10] by Damian. In the setting of Legendrian contact homology with twisted coefficients, an exact Lagrangian filling gives rise to an augmentation taking values in the group ring of the fundamental group of the filling. See the work [5] by the authors for more details, where Legendrian contact homology with twisted coefficients is used to study the topology of Lagrangian fillings and cobordisms.

In Section 6 we outline how our construction can be used as an efficient computational tool for distinguishing a Legendrian knot from its Legendrian mirror in the case when there are no augmentations in a commutative algebra. Note that it, in general, is much easier to extract invariants from the A_∞ -algebra compared to the DGA.

Finally, we recall that Legendrian contact homology is not the only place where noncommutative graded algebras appear in symplectic geometry. Another source is cluster homology, a proposed generalisation of Lagrangian Floer homology due to Cornea and Lalonde [9], which is supposed to provide an alternative approach to the A_∞ -structures in Floer homology introduced by Fukaya, Oh, Ohta and Ono [24].

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2. Algebraic preliminaries

In this section, we fix some notations and recall some basic definitions from the theory of modules over (possibly noncommutative) algebras. For more details of this theory, we refer to [14]. We also introduce some notation that will simplify the various formulas for the A_∞ -structures that we will define.

2.1. Bimodules and tensor products

In this paper R will always denote a commutative ring and A will denote a unital algebra over R which is not necessarily commutative. Important examples will be the matrix algebra $M_n(R)$ corresponding to the endomorphisms of the free R -module R^n , and the group ring $R[G]$ of an arbitrary group G .

For R -modules M, N we denote by

$$M \otimes N := M \otimes_R N$$

their tensor product as R -modules. Moreover, if M, N are A - A -bimodules, their (balanced) tensor product is denoted by

$$M \boxtimes N := M \otimes_A N.$$

We recall that the balanced tensor product is the quotient of $M \otimes N$ by $ma \otimes n = m \otimes an$ for all $a \in A, m \in M$ and $n \in N$.

We also remind that a free A - A -bimodule M on a generating set B is an A - A -bimodule M and a map $i : B \rightarrow M$ of sets such that, for any A - A -bimodule N and any map $f : B \rightarrow N$ of sets, there is a unique A - A -bimodule morphism $\bar{f} : M \rightarrow N$ such that $\bar{f} \circ i = f$. The elements of $\mathcal{B} = i(B)$ in M are a basis for M . The free A - A -bimodule with basis \mathcal{B} will often be identified with $\bigoplus_{c \in \mathcal{B}} (A \otimes_R A)$, where the action of A from the left (resp. right) acts by multiplication from the left (resp. right) on the left (resp. right) factor. Elements of M will also be written as linear combinations of elements of the form a_+ca_- with $a_\pm \in A$ and $c \in \mathcal{B}$.

A *grading* of an A - A -bimodule M in the group $\mathbb{Z}/\mathbb{Z}2\mu$ is a direct sum decomposition $M = \bigoplus_{g \in \mathbb{Z}/\mathbb{Z}2\mu} M_g$. If $m \in M_g$, we write $|m| = g$. The tensor product of graded bimodules is graded by the usual rule.

2.2. Tensor algebras

Given an A - A -bimodule M , we define the *tensor algebra* of M as the algebra

$$\mathcal{T}_A(M) := \bigoplus_{n=0}^{\infty} M^{\boxtimes n}$$

with the multiplication

$$\begin{aligned} \mathfrak{m}: \mathcal{T}_A(M) \boxtimes \mathcal{T}_A(M) &\rightarrow \mathcal{T}_A(M), \\ \mathfrak{m}((m_1 \boxtimes \dots \boxtimes m_i) \boxtimes (n_1 \boxtimes \dots \boxtimes n_j)) &= m_1 \boxtimes \dots \boxtimes m_i \boxtimes n_1 \boxtimes \dots \boxtimes n_j. \end{aligned}$$

Here we have used the notation

$$\begin{aligned} M^{\boxtimes 0} &:= A, \\ M^{\boxtimes n} &:= \underbrace{M \boxtimes \dots \boxtimes M}_n. \end{aligned}$$

We will call $M^{\boxtimes 0}$ the *zero-length part* of $\mathcal{T}_A(M)$.

If M is a graded bimodule, then the tensor algebra $\mathcal{T}_A(M)$ inherits a grading

$$\mathcal{T}_A(M) = \bigoplus_{g \in \mathbb{Z}/\mathbb{Z}2\mu} \mathcal{T}_A(M)_g$$

by requiring that the zero-length part lives in degree zero; i.e. $M^{\boxtimes 0} \subset \mathcal{T}_A(M)_g$ and

$$\mathfrak{m}: \mathcal{T}_A(M)_{g_1} \boxtimes \mathcal{T}_A(M)_{g_2} \rightarrow \mathcal{T}_A(M)_{g_1+g_2}.$$

In this article, algebra maps will always be unital. Algebra maps $\mathcal{T}_A(M) \rightarrow \mathcal{T}_A(N)$ between tensor algebras over A will always be morphisms of A - A -bimodules, and in particular they will restrict to the identity $A = M^{\boxtimes 0} \rightarrow N^{\boxtimes 0} = A$ on the zero-length parts. On the other hand, algebra maps $\mathcal{T}_A(M) \rightarrow B$ for a general R -algebra B will be R - R -bimodule morphisms, and their restriction to $M^{\boxtimes 0} = A$ induces a unital R -algebra morphism $A \rightarrow B$. Algebra maps defined on $\mathcal{T}_A(M)$ are determined by their restrictions to $M^{\boxtimes 0} = A$ and $M^{\boxtimes 1} = M$.

Note that, as in the case when A is commutative, there is a notion of “free product” of tensor algebras defined by

$$\mathcal{T}_A(M) \star \mathcal{T}_A(N) := \mathcal{T}_A(M \oplus N),$$

which is again a tensor algebra. Moreover, for algebra maps $f_i: \mathcal{A}_i \rightarrow \mathcal{B}_i$, $i = 1, \dots, n$, between tensor algebras $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}_1, \dots, \mathcal{B}_n$, there is a naturally induced algebra map

$$f_1 \star \dots \star f_n: \mathcal{A}_1 \star \dots \star \mathcal{A}_n \rightarrow \mathcal{B}_1 \star \dots \star \mathcal{B}_n$$

between the corresponding free products.

In all our applications, M will be a free A - A -bimodules. In this case, if the elements c_1, \dots, c_m, \dots freely generate the A - A -bimodule M , they also generate the algebra $\mathcal{T}_A(M)$ in the following sense: every element in $M^{\boxtimes n}$ can be written as $a_0 c_{i_1} a_1 \dots c_{i_n} a_n$, with $a_0, \dots, a_n \in A$.

2.3. Duals

For an R -module M , we denote by M^* the dual module $\text{Hom}_R(M, R)$. If M is free with a given finite basis \mathcal{B} , then M^* is again free with a dual basis \mathcal{B}^* induced by \mathcal{B} . We recall that, for any $c \in \mathcal{B}$, the dual basis element c^* is the element of M^* which maps c to $1 \in R$ and any other element of \mathcal{B} to 0. Hence, when the basis is part of the data, we will identify M with M^* by identifying c with c^* for all $c \in \mathcal{B}$. If \mathcal{B} is not finite, the above construction only gives an injection $M \rightarrow M^*$.

Given a R -module map $f : M \rightarrow N$, we denote the adjoint morphism by $f^* : N^* \rightarrow M^*$. Again, if M and N are free with given finite bases, then we denote $f^* : N \rightarrow M$.

Let A be any R -algebra. We can regard A as a nonfree A - A -bimodule over itself. For an A - A -bimodule M we will define $M^\vee := \text{Hom}_{A-A}(M, A)$ in the sense of bimodules. Observe that in general M^\vee only has the structure of an R -module. If M is a free and finitely generated A - A -bimodule with a preferred basis \mathcal{B} , then M^\vee can be identified with a free A -module with the same basis. The correspondence is given by

$$M^\vee \ni \varphi \mapsto \sum_{c \in \mathcal{B}} \varphi(c)c. \quad (1)$$

Again, any morphism $f : M \rightarrow N$ of bimodules gives rise to an adjoint morphism

$$f^\vee : N^\vee \rightarrow M^\vee$$

which, typically, is only a morphism of bimodules.

We define morphisms of R -modules $\psi_n : (M^\vee)^{\otimes n} \rightarrow (M^{\boxtimes n})^\vee$ by

$$\psi_n(\beta_1 \otimes \dots \otimes \beta_n)(m_1 \boxtimes \dots \boxtimes m_n) = \beta_1(m_1) \dots \beta_n(m_n) \quad (2)$$

for $\beta_i \in M^\vee$ and $m_i \in M$. Note that $\psi_n(\beta_1 \otimes \dots \otimes \beta_n)$ is well defined on the balanced tensor product since the β_i are bimodule morphisms. The maps ψ_n cannot be seen as morphisms of bimodules in any sensible way.

If M is graded, then the dual modules $M^* = \text{Hom}_R(M, R)$ and $M^\vee = \text{Hom}_{A-A}(M, A)$ (when defined) are also graded with gradings $(M^\vee)_g := (M_{g-1})^\vee$ and $(M^*)_g := (M_{g-1})^*$, i.e. the *suspension* of the dual gradings.

2.4. Hermitian algebras

We are interested in duals $\text{Hom}_R(M, R)$ of A - A -bimodules M for algebras A which are not necessarily finitely generated free R -modules. For that reason, in order to have a better behaving theory, we will introduce some additional structure on the algebra A .

A commutative ring R is an *involutive ring* if it is endowed with an involution $r \mapsto \bar{r}$ (called *conjugation*), which is also a ring homomorphism. The prototypical example to keep in mind is the field of complex numbers, but we will also allow involutive rings where the conjugation is the identity. From now on every ring will be tacitly considered involutive, possibly with a trivial involution.

Definition 2.1. A *Hermitian algebra* (A, \star, \natural) over an involutive ring R consists of:

- an R -algebra A ,

- a map

$$\begin{aligned} \star: A &\rightarrow A, \\ a &\mapsto a^\star, \end{aligned}$$

satisfying

- (1) $(ra + sb)^\star = \bar{r}a^\star + \bar{s}b^\star$ for all $r, s \in R$ and all $a, b \in A$,
- (2) $(ab)^\star = b^\star a^\star$, and
- (3) $(a^\star)^\star = a$, and

- a Hermitian form

$$\mathfrak{t}: A \times A \rightarrow R$$

such that

$$\mathfrak{t}(ba, c) = \mathfrak{t}(a, b^\star c) = \mathfrak{t}(b, ca^\star)$$

for all $a, b, c \in A$, and which is non-degenerate in the following strong sense. For any $n \geq 1$, the morphism

$$\begin{aligned} \iota: \underbrace{A \otimes_R \dots \otimes_R A}_n &\rightarrow \underbrace{(A \otimes_R \dots \otimes_R A)^\star}_n, \\ x &\mapsto \iota_x, \end{aligned}$$

determined by

$$\iota_{a_1 \otimes \dots \otimes a_n}(a'_1 \otimes \dots \otimes a'_n) = \mathfrak{t}(a'_1, a_1) \cdot \dots \cdot \mathfrak{t}(a'_n, a_n) \in R$$

is injective.

From (2) and (3) it follows that $1^\star = 1$. In fact $a = (a^\star)^\star = (1a^\star)^\star = a1^\star$, and similarly $a = 1^\star a$ for all $a \in A$.

Remark 2.2. There are two cases in which the above non-degeneracy for $n > 1$ follows from the case $n = 1$:

- (1) A is free (possibly infinitely generated) as an R -module and \mathfrak{t} is induced by the canonical pairing of its basis elements, or
- (2) R is a domain.

Note that, if the conjugation on R is trivial, \mathfrak{t} is a symmetric bilinear form.

Our main examples of Hermitian algebras will be the group ring $R[G]$ over an arbitrary group G and the matrix algebras $M_n(R)$; in both cases R is an arbitrary commutative ring. On the group ring the involution is induced by the inverse in G , i.e. $g^\star = g^{-1}$ on the basis elements $g \in G$, and \mathfrak{t} is the scalar product for which the group elements $g \in G$ form an orthonormal basis. On the matrix algebra $M_n(R)$ we distinguish whether the conjugation on R is trivial or not. In the first case, the involution in $M_n(R)$ is the transposition, and in the second case it is the adjoint (i.e. the transposition followed by the conjugations). In both types of matrix algebras,

$$\mathfrak{t}(a, b) := \text{tr}(b^\star a), \quad a, b \in M_n(R)$$

is given by the trace. For simplicity, from now on we will consider only Hermitian algebras over commutative rings whose conjugation is trivial.

2.5. Bimodules over Hermitian algebras and their duals

Let M be an arbitrary A - A -bimodule over a Hermitian algebra A . The involution on A allows us to define an A - A -bimodule structure on $M^* = \text{Hom}_R(M, R)$ by

$$(a_1 \varphi a_2)(m) := \varphi(a_1^* m a_2^*) \quad (3)$$

for any $a_1, a_2 \in A$, $m \in M$ and $\varphi \in \text{Hom}_R(M, R)$.

Lemma 2.3. *Let A be a Hermitian algebra and let $f: M \rightarrow N$ be a morphism between A - A -bimodules M and N . Then the adjoint map $f^*: N^* \rightarrow M^*$ is also a morphism of A - A -bimodules.*

Proof. The proof is a simple computation: Let $\varphi \in N^*$, $m \in M$ and $a_+, a_- \in A$. Then

$$f^*(a_+ \varphi a_-)(m) = \varphi(a_+^* f(m) a_-^*) = \varphi(f(a_+^* m a_-^*)) = a_+ f^*(\varphi(m)) a_-.$$

□

Now consider a free A - A -bimodule M with a preferred basis \mathcal{B} . The bilinear pairing \mathfrak{t} on A and the basis \mathcal{B} induce an R -bilinear pairing on each A - A -bimodule $M^{\boxtimes n}$, $n \geq 0$, which, on elements of the form $a_0 d_1 a_1 \dots d_n a_n$ with $a_i \in A$ and $d_i \in \mathcal{B}$ is defined by

$$\langle a_0 d_1 a_1 \dots d_n a_n, a'_0 d'_1 a'_1 \dots d'_n a'_n \rangle = \begin{cases} \mathfrak{t}(a_0, a'_0) \cdot \dots \cdot \mathfrak{t}(a_n, a'_n) & \text{if } d_i = d'_i, i = 1, \dots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $d'_i \in \mathcal{B}$ as well. It can be checked explicitly that

$$\langle a_1 m a_2, n \rangle = \langle m, a_1^* n a_2^* \rangle$$

holds for all $m, n \in M^{\boxtimes n}$ and $a_1, a_2 \in A$. By the assumption of nondegeneracy of \mathfrak{t} , this pairing then induces an injection

$$\begin{aligned} \iota^{(n)}: M^{\boxtimes n} &\hookrightarrow (M^{\boxtimes n})^*, \\ m &\mapsto \langle m, _ \rangle \in \text{Hom}_R(M^{\boxtimes n}, R), \end{aligned} \quad (4)$$

for each $n \geq 0$. The identifications $\iota^{(n)}$ also satisfy the following property.

Lemma 2.4. *Let A be a Hermitian algebra and let M be a free A - A -bimodule with a preferred basis. Then the inclusion $\iota^{(n)}$ for $n \geq 1$ is a morphism of A - A -bimodules for the bimodule structure on $(M^{\boxtimes n})^*$ described in Equation (3).*

Proof. For any $m \in M$ and $a_1, a_2 \in A$ we compute

$$a_1 \langle m, _ \rangle a_2 = \langle m, a_1^* _ a_2^* \rangle = \langle a_1 m a_2, _ \rangle,$$

which shows the claim. □

We will often tacitly identify $M^{\boxtimes n}$ with its image in $(M^{\boxtimes n})^*$ under the inclusion $\iota^{(n)}$. In general, it is not necessarily the case that an R -module morphism $f: M \rightarrow N$ between free infinitely generated R -modules has an adjoint morphism $f^*: N^* \rightarrow M^*$ that restricts to a morphism $f^*: N \rightarrow M \subset M^*$ for these submodules $N \subset N^*$ and $M \subset M^*$. However, this turns out to be the case for a large class of maps that we are interested in here. First we give the following useful formula.

Lemma 2.5. *Let A be a Hermitian algebra, and let M, N be free finitely generated A - A -bimodules with preferred bases. Consider a morphism $g: M \rightarrow N^{\boxtimes n}$, $n \geq 1$, of A - A -bimodules which vanishes on all basis elements except a single $c \in M$, on which it takes the form*

$$g(c) = a_0 d_1 a_1 \dots d_n a_n,$$

where $d_i \in N$ again are basis elements and $a_i \in A$. Then

$$\begin{aligned} g^*(\langle a'_0 d'_1 a'_1 \dots a'_{n-1} d'_n a'_n, _ \rangle) &= \\ &= \begin{cases} \langle \mathfrak{t}(a_1, a'_1) \dots \mathfrak{t}(a_{n-1}, a'_{n-1}) \rangle \cdot a'_0 a_0^* \cdot c \cdot a_n^* a'_n, _ \rangle, & \text{if } d_i = d'_i, i = 1, \dots, n, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

for arbitrary basis elements $d'_i \in N$ and $a'_i \in A$.

Proof. This is a simple verification; for a generator d we have

$$g^*(\langle a'_0 d'_1 a'_1 \dots a'_{n-1} d'_n a'_n, _ \rangle)(d) = \langle a'_0 d'_1 a'_1 \dots a'_{n-1} d'_n a'_n, g(d) \rangle.$$

This can be nonzero only if $d = c$ where we get:

$$\begin{aligned} \langle a'_0 d'_1 a'_1 \dots a'_{n-1} d'_n a'_n, g(c) \rangle &= \\ &= \begin{cases} \langle \mathfrak{t}(a_0, a'_0) \cdot \mathfrak{t}(a_1, a'_1) \dots \mathfrak{t}(a_n, a'_n) \rangle & \text{if } d_i = d'_i, i = 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand the expression

$$\langle (\mathfrak{t}(a_1, a'_1) \dots \mathfrak{t}(a_{n-1}, a'_{n-1})) \cdot a'_0 a_0^* \cdot c \cdot a_n^* a'_n, d \rangle$$

is nonzero only when $d = c$. Moreover,

$$\langle (\mathfrak{t}(a_1, a'_1) \dots \mathfrak{t}(a_{n-1}, a'_{n-1})) \cdot a'_0 a_0^* \cdot c \cdot a_n^* a'_n, c \rangle = \mathfrak{t}(a_0, a'_0) \cdot \mathfrak{t}(a_1, a'_1) \dots \mathfrak{t}(a_n, a'_n)$$

holds since $\mathfrak{t}(a, b) = \mathfrak{t}(ab^*, 1)$. This concludes the proof. \square

We similarly compute the following relation.

Lemma 2.6. *Let A be a Hermitian algebra, and let M, N be free finitely generated A - A -bimodules with preferred bases. Consider a morphism $g: M \rightarrow N^{\boxtimes n}$, $n \geq 1$, of A - A -bimodules of as in Lemma 2.5. For the morphism*

$$G = \text{Id}_M^{\boxtimes k} \boxtimes g \boxtimes \text{Id}_M^{\boxtimes l}: M^{\boxtimes(k+1+l)} \rightarrow M^{\boxtimes k} \boxtimes N^{\boxtimes n} \boxtimes M^{\boxtimes l}$$

we then have

$$\begin{aligned} G^*(\langle x \boxtimes a'_0 d'_1 a'_1 \dots a'_{n-1} d'_n a'_n \boxtimes y, _ \rangle) &= \\ &= \begin{cases} \langle (\mathfrak{t}(a_1, a'_1) \cdot \dots \cdot \mathfrak{t}(a_{n-1}, a'_{n-1})) \cdot x \boxtimes a'_0 a_0^* \cdot c \cdot a_n^* a'_n \boxtimes y, _ \rangle, & \text{if } d_i = d'_i, i = 1, \dots, n, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

for arbitrary basis elements $d'_i \in N$ and $a'_i \in A$, and any $x \in M^{\boxtimes k}$, $y \in M^{\boxtimes l}$.

Proof. Again it is a matter of checking that $(\mathfrak{t}(a_1, a'_1) \cdot \dots \cdot \mathfrak{t}(a_{n-1}, a'_{n-1})) \cdot x \boxtimes a'_0 a_0^* \cdot c \cdot a_n^* a'_n \boxtimes y$ represents the dual $G^*(\langle x \boxtimes a'_0 d'_1 a'_1 \dots a'_{n-1} d'_n a'_n \boxtimes y, _ \rangle)$. This follows by a straightforward computation similarly to the proof of Lemma 2.5. \square

Proposition 2.7. *Let A be a Hermitian algebra and let M, N be free finitely generated A - A -bimodules with preferred bases. Given a morphism $f: M \rightarrow N^{\boxtimes n}$ of bimodules where $n > 0$, for any $k, l \geq 0$ we consider the induced bimodule morphism*

$$F = \text{Id}_M^{\boxtimes k} \boxtimes f \boxtimes \text{Id}_M^{\boxtimes l}: M^{\boxtimes(k+1+l)} \rightarrow M^{\boxtimes k} \boxtimes N^{\boxtimes n} \boxtimes M^{\boxtimes l}.$$

Then the adjoint

$$F^*: (M^{\boxtimes k} \boxtimes N^{\boxtimes n} \boxtimes M^{\boxtimes l})^* \rightarrow (M^{\boxtimes(k+1+l)})^*$$

is a morphism of A - A -bimodules restricting to a morphism of the form

$$F^*: M^{\boxtimes k} \boxtimes N^{\boxtimes n} \boxtimes M^{\boxtimes l} \rightarrow M^{\boxtimes(k+l+1)},$$

$$F^* = \text{Id}_M^{\boxtimes k} \boxtimes f^* \boxtimes \text{Id}_M^{\boxtimes l},$$

on the submodules defined by the inclusion in Equation (4).

Proof. The fact that F^* is a morphism of A - A -bimodules follows from Lemma 2.3. The latter statement follows directly from Lemma 2.6. Namely, the morphism F considered here can be written as a finite sum of morphisms $\text{Id}_M^{\boxtimes k} \boxtimes g \boxtimes \text{Id}_M^{\boxtimes l}$ satisfying the assumptions of Lemma 2.6. \square

Remark 2.8. Note that in order for property (2) above to hold, it is crucial that $n > 0$. For instance, the property is not satisfied for the adjoint m^* of the multiplication map $m: R[G] \otimes R[G] \rightarrow R[G]$ for the group ring of an infinite group.

3. Differential graded algebras over noncommutative rings

In this section we recall some facts about differential graded algebras which are well known for commutative coefficient rings.

3.1. Definitions

Let R be a unital commutative ring and A a (not necessarily commutative) unital algebra over R .

Definition 3.1. A *differential graded algebra* (\mathcal{A}, ∂) over A is a unital $\mathbb{Z}/2\mu\mathbb{Z}$ -graded algebra \mathcal{A} over A whose differential $\partial: \mathcal{A} \rightarrow \mathcal{A}$ is a morphism of A - A -bimodules satisfying the following properties:

- (1) $\partial \circ \partial = 0$,
- (2) ∂ has degree -1 , and
- (3) $\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial(y)$ for all homogeneous elements $x, y \in \mathcal{A}$.

The last equality is known as *graded Leibniz rule*, and tells us that ∂ is a derivation. Above $|x| \in \mathbb{Z}/2\mu\mathbb{Z}$ is the degree of x . The graded Leibniz rule (and the fact that 1 is homogeneous of degree 0) implies that $\partial(1) = 0$. In fact, $\partial(1) = \partial(1 \cdot 1) = \partial(1) \cdot 1 + 1 \cdot \partial(1) = \partial(1) + \partial(1)$. Since ∂ is a morphism of A - A -bimodules, this implies that $\partial(a \cdot 1) = 0$ for all $a \in A$.

In this article we will consider only “semifree” differential graded algebras with finitely many generators, i.e. whose underlying algebra is the tensor algebra

$$\mathcal{A} = \mathcal{T}_A(M) = \bigoplus_{n=0}^{\infty} M^{\boxtimes n}$$

where M is a finitely generated graded free A - A -bimodule. Moreover we will always assume that M comes with a specified finite basis $\mathcal{B} = \{c_1, \dots, c_k\}$ over A consisting of homogeneous elements.

The differential is determined by its action on M , where it decomposes as

$$\partial|_M = \partial_0 + \partial_1 + \partial_2 + \dots, \quad (5)$$

where $\partial_n: M \rightarrow M^{\boxtimes n}$. Clearly $\partial_i = 0$ for i sufficiently large because M is finitely generated. We refer to $\partial_0: M = M^{\boxtimes 1} \rightarrow M^{\boxtimes 0} = A$ as the *constant part* of ∂ . From the Leibniz rule it follows that the differential of an element $x = a_0 c_{i_1} a_1 \dots c_{i_n} a_n \in M^{\boxtimes n}$ is

$$\begin{aligned} \partial x &= \sum_{j=0}^{n-1} (-1)^{|c_{i_1}| + \dots + |c_{i_{j-1}}|} a_0 c_{i_1} a_1 \dots a_{j-1} \partial(c_{i_j}) a_j \dots c_{i_n} a_n \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} (-1)^{|c_{i_1}| + \dots + |c_{i_{j-1}}|} a_0 c_{i_1} a_1 \dots a_{j-1} \partial_k(c_{i_j}) a_j \dots c_{i_n} a_n. \end{aligned} \quad (6)$$

Combining this with the fact that $\partial \circ \partial = 0$, we get the following relation for the maps ∂_i :

$$\sum_{\substack{k+l-1=n \\ k>0, l \geq 0}} \sum_{i=0}^{k-1} (\sigma^{\boxtimes i} \boxtimes \partial_l \boxtimes \text{Id}^{\boxtimes(k-1-i)}) \circ \partial_k = 0, \quad (7)$$

for any fixed $n \geq 0$. Here σ is the automorphism of M which maps a homogenous element m to $(-1)^{|m|}m$, and $\sigma^0 := 1$.

3.2. Changing the coefficients

Assume that we are given a differential graded algebra with coefficients in A and a morphism $A \rightarrow B$ of unital R -algebras. In certain situations it will be useful to consider a change of coefficients from A to B . We recall that B has an induced structure of an A - A -bimodule and also that $M_B := B \boxtimes M \boxtimes B$ is a free B - B -bimodule for any free A - A -bimodule M .

Lemma 3.2. *Let (\mathcal{A}, ∂) be a semi-free differential graded algebra over A such that \mathcal{A} is isomorphic to $\mathcal{T}_A(M)$ as an algebra, and let $f : A \rightarrow B$ be a unital algebra morphism. Then there exist:*

- a unique semi-free differential graded algebra $(\mathcal{A}_B, \partial_B)$ over B such that \mathcal{A}_B is isomorphic to $\mathcal{T}_B(M_B)$ as an algebra, and
- a unique morphism $\hat{f} : \mathcal{A} \rightarrow \mathcal{A}_B$ of unital graded algebras

satisfying the following properties:

- (1) \hat{f} is the natural morphism of unital R -algebras defined uniquely by the requirements that it restricts to f on $M^{\boxtimes 0} = A$, and induces a graded bijection between the generators of \mathcal{A} and \mathcal{A}_B , and
- (2) $\hat{f} \circ \partial = \partial_B \circ \hat{f}$,

i.e. \hat{f} is a unital DGA morphism.

Proof. The existence of the algebra morphism \hat{f} is immediate. The differential ∂_B on \mathcal{A}_B is defined on the image $\hat{f}(m) \in M_B$ of $m \in M$ to take the value

$$\partial_B(\hat{f}(m)) = \hat{f}(\partial(m)).$$

Since \hat{f} is surjective on the generators of \mathcal{A} this determines ∂_B uniquely after extending ∂_B using the graded Leibniz rule. Using the Leibniz rule, the fact that \hat{f} is an algebra morphism implies that

$$\hat{f} \circ \partial = \partial_B \circ \hat{f}$$

is satisfied on all of \mathcal{A} .

It remains to check that $\partial_B^2 = 0$. Since we clearly have

$$\partial_B^2 \circ \hat{f} = \hat{f} \circ \partial^2 = 0,$$

the fact that \hat{f} is a bijection on the generators implies that $\partial_B^2 = 0$ is satisfied on all of \mathcal{A}_B . \square

The following changes of coefficients was used in the construction of the augmentation category in [2], and will also be relevant in this article. Consider the unital R -algebra $A_n := A \otimes_R (\bigoplus_{i=1}^n Re_i)$, where $\bigoplus_{i=1}^n Re_i$ has the ring structure induced by termwise multiplication, i.e. $e_i \cdot e_j = \delta_{ij}e_i$ and $\sum e_i = 1$. The morphism $f : A \rightarrow A_n$ will be the canonical morphism induced by the above tensor product with $\bigoplus_{i=1}^n Re_i$, i.e. for which $f(1) = e_1 + e_2 + \dots + e_n$.

3.3. Augmentations and linearisations

The graded Leibniz rule is invariant under conjugation by degree-preserving unital algebra automorphisms. More precisely, let $\phi : \mathcal{A} \rightarrow \mathcal{A}$ be such an automorphism; then $\partial_\phi := \phi^{-1} \circ \partial \circ \phi$ is a differential and $(\mathcal{A}, \partial_\phi)$ is again a differential graded algebra. We denote by Π_0 the projection of \mathcal{A} to the zero length part $M^{\boxtimes 0} = A$. The constant part of ∂_ϕ is given by $\Pi_0 \circ \phi^{-1} \circ \partial \circ \phi$. In particular, if the map $\varepsilon := \Pi_0 \circ \phi^{-1}$ satisfies $\varepsilon \circ \partial = 0$, this constant term vanishes. This motivates the definition of an augmentation:

Definition 3.3. Let B be a unital R -algebra together with a unital algebra morphism $f : A \rightarrow B$ of R -modules. An *augmentation* of (\mathcal{A}, ∂) into B is a unital DGA morphism $\varepsilon : \mathcal{A} \rightarrow B$ for which $\varepsilon|_{M^{\boxtimes 0}} = f$. Here B is regarded as a differential graded algebra with trivial differential and concentrated in degree zero. (Therefore $\varepsilon \circ \partial = 0$.)

Note that, in particular, ε is a morphism of A - A -bimodules for the A - A -bimodule structure on B induced by f .

In [6], Chekanov described a linearisation procedure which uses an augmentation to produce a differential on the graded A -module M . While Chekanov originally defined linearisation for differential graded algebra over a commutative ring, it is known that his construction works equally well for differential graded algebras with noncommutative coefficients. We now recall this construction.

From a differential graded algebra (\mathcal{A}, ∂) together with an augmentation $\varepsilon : \mathcal{A} \rightarrow B$ we produce a new differential graded algebra as follows. By applying Lemma 3.2 to $\varepsilon|_{M^{\boxtimes 0}} = f : A \rightarrow B$ we obtain a differential graded algebra $(\mathcal{A}_B, \partial_B)$ with coefficients in B , and using the unital DGA morphism $\hat{f} : \mathcal{A} \rightarrow \mathcal{A}_B$ we define an augmentation $\varepsilon_B : \mathcal{A}_B \rightarrow B$ by the requirement that $\varepsilon_B \circ \hat{f} = \varepsilon$ holds on the generators. Using ε_B we define a unital algebra automorphism $\Phi_\varepsilon : \mathcal{A}_B \rightarrow \mathcal{A}_B$ determined by

$$\Phi_\varepsilon(m) = m + \varepsilon_B(m), \quad m \in M_B.$$

We obtain a differential via the conjugation

$$\partial^\varepsilon := \Phi_\varepsilon \circ \partial_B \circ \Phi_\varepsilon^{-1}.$$

Let $\Pi_0 : \mathcal{A}_B \rightarrow M_B^{\boxtimes 0} = B$ be the natural projection; then it follows that $\Pi_0 \circ \partial_B^\varepsilon = \varepsilon_B \circ \partial_B = 0$. The differential graded algebra $(\mathcal{A}_B, \partial^\varepsilon)$ will be said to be obtained from (\mathcal{A}, ∂) by *developing with respect to the augmentation* ε .

The fact that $(\partial_B^\varepsilon)_0 = 0$ will be important in the next section. Using this, Equation (7) can be rewritten as

$$\sum_{\substack{k+l-1=n \\ k,l>0}} \sum_{i=0}^{k-1} (\sigma^{\boxtimes i} \boxtimes (\partial_B^\varepsilon)_l \boxtimes \text{Id}^{\boxtimes (k-1-i)}) \circ (\partial_B^\varepsilon)_k = 0, \quad (8)$$

for any fixed $n > 0$.

3.4. The free n-copy DGA

Let (\mathcal{A}, ∂) be a differential graded algebra. We consider algebras $\mathcal{A}_{ij} = \mathcal{A}$ for $0 \leq i, j \leq n$ and form a differential graded algebra $(\mathfrak{A}_n, \mathfrak{d})$ where \mathfrak{A}_n is, as a graded algebra, the free product

$$\mathfrak{A}_n := \star_{0 \leq i, j \leq n} \mathcal{A}_{ij},$$

and the differential \mathfrak{d} is induced by ∂ as follows. If c is a generator of \mathcal{A} , we denote by c^{ij} the generator in \mathfrak{A} which corresponds to the copy of c in \mathcal{A}_{ij} . Then

- $\mathfrak{d}_0(c^{ij}) = \partial_0(c) \in A$ if $i = j$, and $\mathfrak{d}_0(c^{ij}) = 0$ if $i \neq j$, and
- the coefficient of $a_0 d_1^{i_1 j_1} a_1 d_2^{i_2 j_2} \dots d_n^{i_n j_n} a_n$ in $\mathfrak{d}(c^{ij})$ is equal to the coefficient of $a_0 d_1 a_1 d_2 \dots d_n a_n$ in $\partial(c)$, where d_i is a sequence of generators, given that $i_1 = i$, $j_n = j$, and $j_{k-1} = i_k$ are satisfied, while this coefficient otherwise vanishes.

As usual, we extend \mathfrak{d} to the whole algebra \mathfrak{A}_n via the graded Leibniz rule. (Since we have not proved yet that $\mathfrak{d}^2 = 0$, strictly speaking, $(\mathfrak{A}_n, \mathfrak{d})$ is only a graded algebra with a derivation so far.) A generator c^{ij} will be called *mixed* if $i \neq j$ and *pure* if $i = j$. Observe that \mathfrak{d} preserves the filtration of \mathfrak{A}_n given by the R -submodules spanned by those words containing at least a number $m \geq 0$ of mixed generators.

A word $a_0 d_1^{i_1 j_1} a_1 d_2^{i_2 j_2} \dots d_n^{i_n j_n} a_n$ will be called *composable* if $j_{k-1} = i_k$ for $2 \leq k \leq n$. Words of length zero and one are automatically composable. We define $\mathfrak{A}_n^c \subset \mathfrak{A}_n$ as the sub- A - A -bimodule generated by composable words. It is immediate to verify that \mathfrak{d} restricts to an endomorphism of \mathfrak{A}_n^c .

To prove that $\mathfrak{d}^2 = 0$ we use an alternative definition using the change of coefficients. Recall the ring $A_n = A \otimes (\bigoplus_{i=1}^n Re_i)$ from the end of Section 3.2. For a semi-free differential graded algebra (\mathcal{A}, ∂) we denote by $(\mathcal{A}_{A_n}, \partial_{A_n})$ the differential graded algebra obtained by the change of coefficients from A to A_n using Lemma 3.2.

Recall that there is grading preserving bijection between the sets of generators of the respective algebras \mathcal{A} and \mathcal{A}_{A_n} .

For a generator c of \mathcal{A} we denote $e_i c e_j \in \mathcal{A}_{A_n}$ by c^{ij} . Note that the c^{ij} 's generate \mathcal{A}_{A_n} , albeit not freely. The differential ∂_{A_n} can now be expressed as follows on any generator c . Given that a term $a_0 d_1 a_1 d_2 \dots d_n a_n$ appears in ∂c , the sum

$$\sum_{i_1, j_n} \sum_{j_{k-1}=i_k} a_0 e_{i_1} d_1 e_{j_1} a_1 d_2^{i_2 j_2} \dots d_{n-1}^{i_{n-1} j_{n-1}} a_{n-1} e_{i_n} d_n e_{j_n} a_n$$

appears in the expression $\partial_{A_n} c$. (Recall here that the e_i are in the centre of A .) This means that the sum

$$\sum_{j_{k-1}=i_k} e_{i_1} a_0 d_1^{i_1 j_1} a_1 d_2^{i_2 j_2} \dots d_{n-1}^{i_{n-1} j_{n-1}} a_{n-1} d_n^{i_n j_n} e_{j_n} a_n$$

appears in the expression $\partial_{A_n}(c^{i_1 j_n})$. More generally, we have $\partial_{A_n}(c^{i_1 j_n}) \subset e_{i_1} \mathcal{A}_{A_n} e_{j_n}$. Since $e_i e_j = \delta_{ij} e_i$, in particular it follows that $\partial_{A_n} c^{ij}$ has no constant term for $i \neq j$.

Lemma 3.4. *We have $\mathfrak{d}^2 = 0$.*

Proof. The first observation is that it is enough to prove that $\mathfrak{d}^2(c^{ij}) = 0$ for every generator c^{ij} of \mathfrak{A}_n . There is an algebra morphism $\pi: \mathfrak{A}_n \rightarrow \mathcal{A}_{A_n}$ such that $\pi(c^{ij}) = e_i c e_j$ and $\pi(1) = e_1 + \dots + e_n$. It is easy to check that $\pi \circ \mathfrak{d} = \partial_{A_n} \circ \pi$, and moreover π is injective on the sub-bimodule \mathfrak{A}_n^c generated by composable words. Since $\mathfrak{d}^2(c^{ij}) \in \mathfrak{A}_n^c$ for every generator c^{ij} , from $\partial_{A_n}^2 = 0$ it follows that $\mathfrak{d}^2 = 0$. \square

From a sequence $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ of augmentations $\varepsilon_i: \mathcal{A} \rightarrow A$ we define an algebra morphism $\mathfrak{e}: \mathfrak{A}_n \rightarrow A$ such that, on a generator c^{ij} of \mathfrak{A}_{ij} ,

$$\mathfrak{e}(c^{ij}) = \begin{cases} \varepsilon_i(c) & \text{if } i = j, \text{ and} \\ 0 & \text{if } i \neq j. \end{cases}$$

Lemma 3.5. *The algebra morphism $\mathfrak{e}: \mathfrak{A}_n \rightarrow A$ is an augmentation.*

Proof. It suffices to check that $\mathfrak{e}(\mathfrak{d}(c^{ij})) = 0$ holds on the generators. By construction, we have

$$\mathfrak{e}(\mathfrak{d}(c^{ij})) = \begin{cases} \varepsilon_i(\partial(c)) & \text{if } i = j, \text{ and} \\ 0 & \text{if } i \neq j, \end{cases}$$

which establishes the claim. \square

We can use the augmentation \mathfrak{e} to produce a differential graded algebra $(\mathfrak{A}_n, \mathfrak{d}^\mathfrak{e})$ whose differential has vanishing constant term by applying the procedure described in the previous subsection.

4. A_∞ operations

4.1. Case I: coefficients in a general noncommutative algebra

Let (\mathcal{A}, ∂) be a differential graded algebra with coefficients in a noncommutative algebra A over a commutative ring R . We further assume that $\mathcal{A} = \mathcal{T}_A(M)$ is a tensor algebra over a free A - A -bimodule M with a preferred basis $\{c_1, \dots, c_k\}$.

Recall that we decompose $\partial|_M = \partial_0 + \partial_1 + \dots$, where $\partial_n: M \rightarrow M^{\boxtimes n}$. We start without the assumption that $\partial_0 = 0$. There are induced adjoints $(\partial_n)^\vee: (M^{\boxtimes n})^\vee \rightarrow M^\vee$, and we define

$$\mu_n := (\partial_n)^\vee \circ \psi_n: (M^\vee)^{\otimes n} \rightarrow M^\vee, \quad n \geq 1, \quad (9)$$

where ψ_n is as defined in Section 2.3 above. See Diagram (10).

$$\begin{array}{ccc} M^\vee & \xleftarrow{(\partial_n)^\vee} & (M^{\boxtimes n})^\vee \\ \psi_1 = \text{Id}_{M^\vee} \parallel & & \uparrow \psi_n \\ M^\vee & \xleftarrow{\mu_n} & (M^\vee)^{\otimes n} \end{array} \quad (10)$$

Given elements $m_1, \dots, m_n \in M^\vee$, we will write $\mu_n(m_1, \dots, m_n)$ and $\mu_n(m_1 \otimes \dots \otimes m_n)$ interchangeably.

The operations μ_n can be expressed more concretely as follows.

Lemma 4.1. *If, for every element c_i in the basis of M ,*

$$\partial_n c_i = \sum_I \sum_{j=1}^{m_{i,I}} a_{j,0}^{i,I} c_{i_1} a_{j,1}^{i,I} \dots c_{i_n} a_{j,n}^{i,I}$$

with $a_{j,l}^{i,I} \in A$, and $I = (i_1, \dots, i_n)$ denoting a multi-index with $1 \leq i_l \leq k$, then

$$\mu_n(b_1 c_{i_1}, \dots, b_n c_{i_n}) = \sum_{i=1}^k \sum_{j=1}^{m_{i,I}} (a_{j,0}^{i,I} b_1 a_{j,1}^{i,I} \dots a_{j,n-1}^{i,I} b_n a_{j,n}^{i,I}) \cdot c_i$$

for each $n \geq 1$ and any elements $b_i \in A$.

Proof. From Equations (1) and (2) it follows that

$$\begin{aligned} & \psi_n(b_1 c_{i_1} \otimes \dots \otimes b_n c_{i_n})(a_0 c_{j_1} a_1 \dots a_{n-1} c_{j_n} a_n) = \\ & = \begin{cases} a_0 b_1 a_1 \dots a_{n-1} b_n a_n, & \text{if } c_{i_1} = c_{j_1}, \dots, c_{i_n} = c_{j_n}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then from

$$\mu_n(b_1 c_{i_1}, \dots, b_n c_{i_n})(c_i) = \psi_n(b_1 c_{i_1} \otimes \dots \otimes b_n c_{i_n})(\partial_n c_i)$$

for all basis elements c_i , and from Equation (1) again, the lemma follows. \square

Remark 4.2. The maps μ_n are R -multilinear, but do not satisfy any form of A -linearity in general.

The maps $(\mu_i)_{i \geq 1}$ do not necessarily satisfy the A_∞ relations because the curvature term $\mu_0 := (\partial_0)^\vee: A \rightarrow M^\vee$ might be non-vanishing. Given augmentations of \mathcal{A} , this can be amended.

To an augmentation $\varepsilon: \mathcal{A} \rightarrow A$ we associate the element

$$\varepsilon^\vee := \varepsilon(c_1)c_1 + \dots + \varepsilon(c_k)c_k \in M^\vee,$$

i.e. the adjoint of $\varepsilon|_M$.

Remark 4.3. We should think of ε^\vee as giving rise to a ‘‘bounding cochain’’ in the sense of [24] via the infinite sum

$$\sum_{i=0}^{\infty} (\varepsilon^\vee)^{\otimes i}$$

living in the completion $\prod_{i=0}^{\infty} (M^\vee)^{\otimes i}$ of $\bigoplus_{i=0}^{\infty} (M^\vee)^{\otimes i}$, where $(\varepsilon^\vee)^{\otimes 0} := 1 \in R$.

Definition 4.4. For a sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of augmentations $\varepsilon_i: \mathcal{A} \rightarrow A$, we define the operations

$$\mu_n^\varepsilon: (M^\vee)^{\otimes n} \rightarrow M^\vee$$

via the formulas

$$\begin{aligned} \mu_n^\varepsilon(m_1, \dots, m_n) &= \\ &= \sum_{i=1}^{\infty} \sum_{\substack{i_0 + \dots + i_n + n = i \\ i_j \geq 0}} \mu_i((\varepsilon_0^\vee)^{\otimes i_0} \otimes m_1 \otimes (\varepsilon_1^\vee)^{\otimes i_1} \otimes \dots \otimes (\varepsilon_{n-1}^\vee)^{\otimes i_{n-1}} \otimes m_n \otimes (\varepsilon_n^\vee)^{\otimes i_n}), \end{aligned}$$

where $m_i \in M^\vee$ (recall that $(\varepsilon_0^\vee)^{\otimes 0} = 1$ is considered as an element in R).

See Figure 1 for a geometric explanation of the terms appearing in $\mu_n^\varepsilon(m_1, \dots, m_n)$.

Given a n -tuple of augmentation $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ we denote by $\varepsilon_{ij} = (\varepsilon_i, \dots, \varepsilon_j)$ and $\hat{\varepsilon}_{ij} = (\varepsilon_1, \dots, \varepsilon_i, \varepsilon_j, \dots, \varepsilon_n)$.

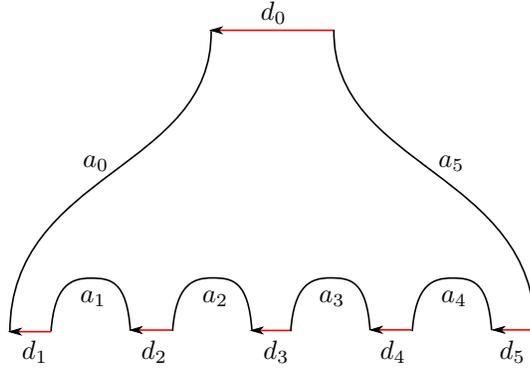


FIGURE 1. The “pseudoholomorphic disk with punctures” shown here is supposed to give a contribution of $+1$ to the coefficient in front of $a_0 d_1 a_1 d_2 a_2 d_3 a_3 d_4 a_4 d_5 a_5$ in the expression $\partial_5(d_0)$. If the generators d_1, \dots, d_5 are all distinct, this pseudoholomorphic disc gives a contribution of $a_0 \varepsilon_0(d_1) a_1 x a_2 \varepsilon_1(d_3) a_3 y a_4 \varepsilon_2(d_4) a_5 \in A$ to the coefficient in front of d_0 in the expression of $\mu_2^{(\varepsilon_0, \varepsilon_1, \varepsilon_2)}(x d_2, y d_4)$.

Theorem 4.5. *For any $n \geq 1$ and fixed sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of augmentations, the above R -module morphisms satisfy the relations*

$$\begin{aligned} 0 &= \sum_{\substack{n=k-1+l \\ k, l > 0}} \sum_{i=1}^k (-1)^\dagger \mu_k^{\hat{\varepsilon}^{i-1, i+l-1}}(m_1, \dots, m_{i-1}, \\ &\quad \mu_l^{\varepsilon^{i-1, i+l-1}}(m_i, \dots, m_{i+l-1}), m_{i+l}, \dots, m_n), \end{aligned}$$

where $\dagger = |m_1| + \dots + |m_{i-1}| + (i-1)$, and $m_1, \dots, m_n \in M^\vee$ are homogeneous elements. In other words, the above operations form the morphisms and higher operations of an A_∞ -category over R whose objects consist of the augmentations $\varepsilon: \mathcal{A} \rightarrow A$.

Proof. First we handle the case when all augmentations are equal and trivial (i.e. sending all generators to zero). In this case the A_∞ -relations readily follow from Formula (8), which is satisfied when $\partial_0 = 0$. Also, see the following diagram.

$$\begin{array}{ccc}
 (M^{\boxtimes k})^\vee & \xleftarrow{(\sum \sigma \boxtimes \partial_t \boxtimes \text{Id})^\vee} & (M^{\boxtimes (k+l)})^\vee \\
 \uparrow \psi_k & & \uparrow \psi_{k+l} \\
 (M^\vee)^{\otimes k} & \xleftarrow{\sum \sigma \otimes \mu_t \otimes \text{Id}} & (M^\vee)^{\otimes (k+l)}.
 \end{array} \tag{11}$$

The general case can now be reduced to the above case in the following manner. First, given a sequence $\varepsilon = (\varepsilon, \dots, \varepsilon)$ consisting of a single augmentation, we compute that $(\mu_i^\varepsilon)_{i \geq 1}$ associated to (\mathcal{A}, ∂) are equal to the morphisms $(\mu_i)_{i \geq 1}$ associated to the DGA $(\mathcal{A}, \partial^\varepsilon)$ defined in Subsection 3.3. This case thus follows from the above.

Finally, for an arbitrary sequence $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ of augmentations we apply the construction in Section 3.4. Namely, we produce the auxiliary “semisimple” differential graded algebra $(\mathfrak{A}_n, \mathfrak{d})$ and the auxiliary augmentation $\mathfrak{e}: \mathfrak{A}_n \rightarrow A$ induced by ε . Using the notation in Section 3.4, it can be seen that

$$\mu_n^\varepsilon(a_1 d_1, \dots, a_n d_n) = \mu_n^\varepsilon(a_1 d_1^{i_1 j_1}, \dots, a_n d_n^{i_n j_n})$$

(after identifying the the output with an element of $\mathcal{A}_{i_1 j_n} = \mathcal{A}$), where $i_k < j_k$ and $j_k = i_{k+1}$ holds for all indices k , and d_i is a sequence of basis elements. We have thus managed to reduce the general case to the first case.

For the sign \dagger it suffices to notice that $(\sigma)^\vee = -\sigma$ according to sign the convention of Section 2.3. \square

4.2. Case II: coefficients in a Hermitian algebra

Let (\mathcal{A}, ∂) be a differential graded algebra with coefficients in a noncommutative algebra A over a commutative ring R . As in the previous subsection, we assume that $\mathcal{A} = \mathcal{T}_A(M)$ is a tensor algebra over an free A - A -bimodule M with a preferred basis $\{c_1, \dots, c_k\}$. In this subsection we make the assumption that A is a Hermitian algebra. Recall that there are induced inclusions $\iota^{(n)}: M^{\boxtimes n} \rightarrow (M^{\boxtimes n})^*$ for each $n \geq 0$ induced by the basis on M and by the bilinear form \mathfrak{t} on A .

We define the A - A -bimodule morphisms

$$\mu_n := (\partial_n)^*: (M^{\boxtimes n})^* \rightarrow (M)^*$$

for each $n \geq 1$. In view of Proposition 2.7 these morphisms restrict to morphisms $\mu_n := (\partial_n)^*: M^{\boxtimes n} \rightarrow M$ under the above inclusions. However, since ∂_0 is not assumed to be zero, these operations might not give rise to an A_∞ structure in the strict sense. We now proceed to amend this.

Using Lemma 2.5, the operations μ_n can in this case be expressed more concretely as follows.

Lemma 4.6. *If, for every element c_i in the basis of M ,*

$$\partial_n c_i = \sum_I \sum_{j=1}^{m_{i,I}} a_{j,0}^{i,I} c_{i_1} a_{j,1}^{i,I} \cdots c_{i_n} a_{j,n}^{i,I}$$

with $a_{j,l}^{i,I} \in A$, and $I = (i_1, \dots, i_n)$ denoting a multi-index with $1 \leq i_l \leq k$, then

$$\begin{aligned} \mu_n(b_0 c_{i_1} b_1 \cdots b_{n-1} c_{i_n} b_n) &= \\ &= \sum_{i=1}^k \sum_{j=1}^{m_{i,I}} \langle c_{i_1} b_1 \cdots b_{n-1} c_{i_n}, c_{i_1} a_{j,1}^{i,I} \cdots a_{j,n-1}^{i,I} c_{i_n} \rangle b_0 (a_{j,0}^{i,I})^* c_i (a_{j,n}^{i,I})^* b_n \end{aligned}$$

for each $n \geq 1$ and any elements $b_i \in A$.

Given an augmentation $\varepsilon: \mathcal{A} \rightarrow A$, we define the adjoints

$$\varepsilon_{(n)}^*: A^* \rightarrow (M^{\boxtimes n})^*$$

for each $n \geq 0$, where $\varepsilon_{(0)}^* = \text{Id}_{A^*}$. Again these maps are related to the notion of a “bounding cochain”. As a side remark, We note that

$$\varepsilon_{(i)}^*(a) = a \cdot \varepsilon_{(i)}^*(1) = \varepsilon_{(i)}^*(1) \cdot a \quad (12)$$

holds for the A - A -bimodule structure defined by (3).

Remark 4.7. When the algebra A is free as an R -module and the pairing \mathfrak{t} is induced by an orthonormal basis, the “bounding cochains”

$$\varepsilon_{(n)}^*: A \rightarrow (M^{\boxtimes n})^*$$

can be expressed as

$$\varepsilon_{(n)}^*(a) = \langle a, \varepsilon(_) \rangle = \sum_{a_0 d_1 a_1 \dots a_{n-1} d_n a_n} \mathfrak{t}(a, \varepsilon(a_0 d_1 a_1 \dots a_{n-1} d_n a_n)) a_0 d_1 a_1 \dots a_{n-1} d_n a_n, \quad (13)$$

where the sum is taken over all words $a_0 d_1 a_1 \dots a_{n-1} d_n a_n$ such that a_1, \dots, a_n are elements of the orthonormal basis of A and d_1, \dots, d_n are elements of the prescribed basis of M (both allowing repetitions).

As a double check we verify Equation 12 for $n = 1$ using Equation (13). If $\{c_1, \dots, c_k\}$ is the basis of M , we denote $\varepsilon_i = \varepsilon(c_i)$. Then Equation (13) for $n = 1$ can be rewritten as

$$\varepsilon_{(1)}^*(a) = \sum_{i=1}^k \sum_{a_+, a_-} \mathfrak{t}(a, a_+ \varepsilon_i a_-) a_+ c_i a_-,$$

where a_+ and a_- run through the orthonormal basis of A . Now we observe that

$$\sum_{a_+} \mathfrak{t}(a, a_+ \varepsilon_i a_-) a_+ = \sum_{a_+} \mathfrak{t}(a a_-^* \varepsilon_i^*, a_+) a_+ = a a_-^* \varepsilon_i^*$$

and therefore we can rewrite

$$\varepsilon_{(1)}^*(a) = \sum_{i=1}^k \sum_{a_-} aa_-^* \varepsilon_i^* c_i a_-.$$

On the other hand we have

$$\begin{aligned} a\varepsilon_{(1)}(1) &= \sum_{i=1}^k \sum_{a_+, a_-} \mathfrak{t}(1, a_+ \varepsilon_i a_-) aa_+ c_i a_- = \\ &= \sum_{i=1}^k \sum_{a_+, a_-} \mathfrak{t}(a_-^* \varepsilon_i^*, a_+) aa_+ c_i a_- = \sum_{i=1}^k \sum_{a_-} aa_-^* \varepsilon_i^* c_i a_-. \end{aligned}$$

Then half of Equation (12) is verified. The other half is similar.

Definition 4.8. Given a sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of augmentations of \mathcal{A} , we define the operations

$$\mu_n^\varepsilon: (M^{\boxtimes n}) \rightarrow M^*, \quad n \geq 1,$$

via the formulas

$$\begin{aligned} \mu_n^\varepsilon(a_0 m_1 a_1 \dots a_{n-1} m_n a_n) &= \\ &= \sum_{i=1}^{\infty} \sum_{\substack{i_0 + \dots + i_n + n = i \\ i_j \geq 0}} \mu_i((\varepsilon_0)_{(i_0)}^*(a_0) \boxtimes m_1 \boxtimes (\varepsilon_1)_{(i_1)}^*(m_1) \boxtimes \dots \boxtimes m_n \boxtimes (\varepsilon_n)_{(i_n)}^*(a_n)), \end{aligned} \quad (14)$$

where $m_1, \dots, m_n \in M$ and $a_0, \dots, a_n \in A$ and $a_0 m_1 a_1 \dots a_{n-1} m_n a_n \in M^{\boxtimes n}$ is identified to an element in $(M^{\boxtimes n})^*$ by the inclusion $\iota_{(n)}: M^{\boxtimes n} \rightarrow (M^{\boxtimes n})^*$ (see Equation (2.5)).

Lemma 4.9. *The compositions in Formula (14) give rise to a well-defined map*

$$\mu_n^\varepsilon: M^{\boxtimes n} \rightarrow M \subset M^*.$$

Proof. Since

$$\mu_n^\varepsilon := \sum_{i=1}^{\infty} \sum_{\substack{i_0 + \dots + i_n + n = i \\ i_j > 0}} (((\varepsilon_0)^{\boxtimes i_0} \boxtimes \text{Id}_M \boxtimes (\varepsilon_1)^{\boxtimes i_1} \boxtimes \dots \boxtimes \text{Id}_M \boxtimes (\varepsilon_n)^{\boxtimes i_n}) \circ \partial_i)^*,$$

the statement follows from Proposition 2.7. \square

Remark 4.10. The operations μ_n^ε are morphisms of A - A -bimodules by Lemma 2.3 and Equation (12).

The main result of this section is that these operations define an A_∞ -category.

Theorem 4.11. *For any $n \geq 1$ and fixed sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of augmentations, the operations in Definition 4.8 satisfy the following A_∞ relations:*

$$\sum_{\substack{n=k-1+l \\ k, l > 0}} \sum_{i=1}^k (-1)^\dagger \mu_k^{\hat{\varepsilon}^{i-1, i+l-1}}(m_1, \dots, m_{i-1}, \mu_l^{\varepsilon^{i-1, i+l-1}}(m_i, \dots, m_{i+l-1}), m_{i+l}, \dots, m_n) = 0,$$

where $\dagger = |m_1| + \dots + |m_{i-1}| + (i-1)$, and $m_1, \dots, m_n \in M$ are of homogeneous degree. In other words, the above operations form the morphisms and higher operations of an A_∞ -category over A whose objects consist of the augmentations $\varepsilon: (\mathcal{A}, \partial) \rightarrow (A, 0)$.

Proof. The proof follows mutatis mutandis from the proof of Theorem 4.5. To that end, we just have to check the fact that the equality

$$\mathrm{Id}^{\boxtimes i-1} \boxtimes \mu_l \boxtimes \mathrm{Id}^{\boxtimes (k-i)} = (\mathrm{Id}^{\boxtimes i-1} \boxtimes \partial_n \boxtimes \mathrm{Id}^{\boxtimes (k-i)})^*$$

is satisfied. □

5. A toy example of a DGA

In this section we discuss a toy example which illustrates the two different A_∞ -structures defined. This example was suggested by Lenny Ng and was an inspiration for this paper.

Let A be an algebra over $R = \mathbb{Z}/2\mathbb{Z}$ and let g_1, g_2 be two elements of A which do not (necessarily) commute. We will consider the differential graded algebra (\mathcal{A}, ∂) over A generated by c_1, \dots, c_5 and with differential

$$\begin{aligned} \partial c_1 &= c_2 g_1 c_4 + c_3, \\ \partial c_2 &= c_5 g_2, \\ \partial c_3 &= c_5 g_2 g_1 c_4, \\ \partial c_4 &= \partial c_5 = 0. \end{aligned}$$

It is easily checked that $\partial^2 = 0$. Moreover, ∂_0 vanishes and therefore there is a canonical augmentation which sends every generator to zero.

5.1. The A_∞ -structure defined in Subsection 4.1 (Case I)

The construction of Subsection 4.1, performed on the trivial augmentation, gives rise to an A_∞ -algebra structure on $M^\vee = \bigoplus_{i=1}^5 A c_i$. Let $a, a' \in A$. Using Lemma 4.1 we compute the first order operations

$$\begin{aligned} \mu_1(ac_1) &= \mu_1(ac_2) = \mu_1(ac_4) = 0, \\ \mu_1(ac_3) &= ac_1, \\ \mu_1(ac_5) &= ag_2 \cdot c_2, \end{aligned}$$

and the second order operations

$$\begin{aligned} \mu_2(ac_2, a'c_4) &= ag_1 a' \cdot c_1, \\ \mu_2(ac_5, a'c_4) &= ag_2 g_1 a' \cdot c_3, \end{aligned}$$

while $\mu_2(ac_i, a'c_j) = 0$ whenever $(i, j) \notin \{(2, 4), (5, 4)\}$. Finally, $\mu_n \equiv 0$ for all $n \geq 3$.

We verify that these operations verify the A_∞ relations. The only nontrivial relation to verify (i.e. the only one where not all terms vanish) is

$$\begin{aligned} & \mu_1(\mu_2(a_1c_5, a_2c_4)) + \mu_2(\mu_1(a_1c_5), a_2c_4) + \mu_2(a_1c_5, \mu_1(a_2c_4)) = \\ & = \mu_1(a_1g_2g_1a_2c_3) + \mu_2(a_1g_2c_2, a_2c_4) + 0 \\ & = a_1g_2g_1a_2c_1 + a_1g_2g_1a_2c_1 \\ & = 0. \end{aligned}$$

5.2. The A_∞ -structure defined in Subsection 4.2 (Case II)

In this subsection we assume that A is a Hermitian algebra. The construction of Subsection 4.2, performed on the trivial augmentation, gives rise to an A_∞ -algebra structure on M , which we identify to a submodule of M^* by (2.5). Elements in $M^{\boxtimes n}$ can be written, as usual, as linear combinations of terms of the form $a_0c_{i_1}a_1 \dots a_{n-1}c_{i_n}a_n$, where $a_0, \dots, a_n \in A$ and c_{i_1}, \dots, c_{i_n} are element of the prescribed basis of M . Since in case II the operations μ_n are morphisms of A - A -bimodule, we will give their values only on elements of the form $c_{i_1}a_1 \dots a_{n-1}c_{i_n}$.

Using Lemma 4.6 we compute the first order operations

$$\begin{aligned} \mu_1(c_1) &= \mu_1(c_2) = \mu_1(c_4) = 0, \\ \mu_1(c_3) &= c_1, \\ \mu_1(c_5) &= c_2g_2^* \end{aligned}$$

and the second order operations, for all $h \in A$,

$$\begin{aligned} \mu_2(c_2hc_4) &= \mathfrak{t}(h, g_1)c_1, \\ \mu_2(c_5hc_4) &= \mathfrak{t}(h, g_2g_1)c_3, \end{aligned}$$

while $\mu_2(c_ihc_j) = 0$ in all other cases. Finally, $\mu_n \equiv 0$ for all $n \geq 3$.

The only nontrivial A_∞ relation to check is

$$\begin{aligned} & \mu_1(\mu_2(c_5hc_4)) + \mu_2(\mu_1(c_5) \boxtimes hc_4) + \mu_2(c_5h \boxtimes \mu_1(c_4)) = \\ & = \mathfrak{t}(h, g_2g_1)\mu_1(c_3) + \mu_2(c_2g_2^*hc_4) + 0 \\ & = \mathfrak{t}(h, g_2g_1)c_1 + \mathfrak{t}(g_2^*h, g_1)c_1 \\ & = 0 \end{aligned}$$

because $\mathfrak{t}(h, g_2g_1) = \mathfrak{t}(g_2^*h, g_1)$ by the properties of the adjoint.

6. Potential examples of knots distinguished by the noncommutative A_∞ -structures

In general it is easier to extract information from the A_∞ -algebra associated to an augmentation than from the Chekanov-Eliashberg algebra. For instance, the products and higher order Massey products in linearised Legendrian cohomology introduced in [8]

were shown in the same article to be efficient tools for distinguishing a Legendrian knot from its mirror (in case when the underlying homologies are isomorphic). This application used the A_∞ -structure with coefficients in \mathbb{Z}_2 .

Assume that there exists a Legendrian knot $\Lambda_n \subset (\mathbb{R}^3, dz - ydx)$ which satisfies the following for some $n > 1$:

- (i) The rotation number of Λ_n is zero;
- (ii) The bound on the Thurston-Bennequin invariant of Λ_n in terms of the Kauffman polynomial of the underlying smooth knot is not sharp. In particular, this means that the Chekanov-Eliashberg algebra of Λ_n has no augmentation in \mathbb{Z}_2 (see [31] for more details); and
- (iii) The Chekanov-Eliashberg algebra of Λ_n admits a (0-graded) augmentation in $M_n(\mathbb{Z}_2)$.

The authors expect that such knots can be constructed using the methods from [28, Theorem 4.8] but, unfortunately, as of now they are not aware of an explicit example. Performing cusp connected sums (see [20]) between the Legendrian knots considered in [8] and the hypothetical Legendrian knot Λ_n we obtain Legendrian knots for which our construction can be used as an efficient computational tool; see Proposition 6.1 below.

First we recall some details concerning the examples from [8]. For a Legendrian knot Λ we denote by $\bar{\Lambda}$ its *Legendrian mirror*, i.e. its image under the contactomorphism

$$(x, y, z) \mapsto (x, -y, -z).$$

Consider the examples $\Lambda_{k,l,m}$ constructed in the proof of [8, Theorem 1.1] (the first part) which satisfy the following. For any triple $k, l, m \geq 1$ it is the case that

- the rotation number of $\Lambda_{k,l,m}$ is zero, and hence the DGA is graded over the integers, and
- if the three numbers $l - m - 1$, $m - k + 1$, $l - k + 1$, are distinct, then there is a unique graded augmentation ε in \mathbb{Z}_2 .

The same properties are satisfied for the Legendrian mirror $\overline{\Lambda_{k,l,m}}$. Given k, l, m satisfying the second property, the knot $\Lambda_{k,l,m}$ is distinguished from its mirror up to Legendrian isotopy by a computation showing that

- (1) the product

$$\mu_2^{\varepsilon, \varepsilon, \varepsilon} : LCH_\varepsilon^{l-m-1}(\Lambda_{k,l,m}) \otimes LCH_\varepsilon^{m-k+1}(\Lambda_{k,l,m}) \rightarrow LCH_\varepsilon^{l-k}(\Lambda_{k,l,m})$$

does not vanish identically, while

- (2) the product

$$\mu_2^{\varepsilon, \varepsilon, \varepsilon} : LCH_\varepsilon^{l-m-1}(\overline{\Lambda_{k,l,m}}) \otimes LCH_\varepsilon^{m-k+1}(\overline{\Lambda_{k,l,m}}) \rightarrow LCH_\varepsilon^{l-k}(\overline{\Lambda_{k,l,m}})$$

does vanish.

We can use these examples to show the following.

Proposition 6.1. *Assume the existence of a Legendrian knot Λ_n for some $n > 1$ satisfying conditions (i)–(iii) above. The cusp connected sum $\Lambda_{k,l,m} \# \Lambda_n \subset (\mathbb{R}^3, dz - ydx)$*

is a Legendrian knot admitting a (0-graded) augmentation in $M_n(\mathbb{Z}_2)$ but not in \mathbb{Z}_2 . For suitable $k, l, m > 0$ (depending on the knot Λ_n) the Legendrian knot $\Lambda_{k,l,m} \# \Lambda_n$ can moreover be distinguished from both $\overline{\Lambda_{k,l,m}} \# \Lambda_n$ and $\overline{\Lambda_{k,l,m}} \# \overline{\Lambda_n}$ using the A_∞ -structure in linearised Legendrian contact homology with coefficients in $M_n(\mathbb{Z}_2)$ as constructed in Section 4.2 (i.e. Case II).

Sketch of proof. The fact that the connected sum has augmentations in $M_n(\mathbb{Z}_2)$ but not in \mathbb{Z}_2 was shown in [13, Lemma 4.3]. The existence part uses the following explicit construction that we now outline. Given augmentations $\varepsilon_i: \Lambda_i \rightarrow A_i$, $i = 1, 2$, in the unital algebras A_i , then there is an induced augmentation $(\varepsilon_1 \# \varepsilon_2): \mathcal{A}(\Lambda_1 \# \Lambda_2) \rightarrow A_1 \otimes A_2$ determined as follows. Recall that

$$\mathcal{A}(\Lambda_1 \# \Lambda_2) = \mathcal{A}(\Lambda_1) \star \mathcal{A}(\Lambda_2) \star \langle c_0 \rangle$$

holds on the level of generators, where $|c_0| = 0$. The induced augmentation is determined uniquely by the requirements that $(\varepsilon_1 \# \varepsilon_2)(c) = \varepsilon_i(c)$ holds on the old generators (using the canonical algebra maps $a \mapsto a \otimes 1_{A_2} \in A_1 \otimes A_2$ and $b \mapsto 1_{A_1} \otimes b \in A_1 \otimes A_2$) while $(\varepsilon_1 \# \varepsilon_2)(c_0) = 1 = 1_{A_1} \otimes 1_{A_2} \in A_1 \otimes A_2$ holds on the new generator.

The computations of the DGA of $\Lambda_{k,l,m}$ performed in [8] can readily be seen to give the following. Consider the construction of the A_∞ -structure with coefficients in $M_n(\mathbb{Z}_2)$ as defined in Section 4.2 (i.e. Case II). The non-vanishing of the product as in (1) again holds for $\Lambda_{k,l,m} \# \Lambda_n$ when using the augmentation $\varepsilon \# \varepsilon_2$ taking values in $M_n(\mathbb{Z}_2)$. It can moreover be seen that that (2) is satisfied for *any* pair of graded augmentations in $M_n(\mathbb{Z}_2)$ for the same coefficients, given that $k, l, m > 0$ were chosen appropriately. E.g. we can choose $k, l, m > 0$ so that $l - m - 1$, $m - k + 1$, $l - k + 1$ all are distinct and sufficiently large (depending on the degrees of the Reeb chords of Λ_n). \square

The following result shows the relation between the linearised Legendrian contact cohomology of a Legendrian knot and its Legendrian mirror.

Lemma 6.2. *Let $\Lambda \subset (\mathbb{R}^3, dz - ydx)$ be a Legendrian knot. For any pair of augmentations $\varepsilon_i: (\mathcal{A}(\Lambda), \partial) \rightarrow M_n(R)$, $i = 1, 2$, there are corresponding induced augmentations $\bar{\varepsilon}_i: (\mathcal{A}(\bar{\Lambda}), \partial') \rightarrow M_n(R)$ for which there is a canonical isomorphism*

$$(LCC^\bullet(\Lambda), d^{\varepsilon_0, \varepsilon_1}) \simeq (LCC^\bullet(\bar{\Lambda}), d^{\bar{\varepsilon}_1, \bar{\varepsilon}_0})$$

of graded R -bimodules. (This can even be made into an isomorphism of free $M_n(R)$ -bimodules, but in this case we must use a non-standard free bimodule structure on the latter where left and right multiplication has been interchanged, while utilising the transpose of a matrix.)

Proof. Recall that there is a canonical grading-preserving bijection between the set of generators of Λ and $\bar{\Lambda}$. Under the corresponding identification $\mathcal{A}(\Lambda) \simeq \mathcal{A}(\bar{\Lambda})$ the differential of the latter takes the form $\partial'(c) = \iota \circ \partial(c)$ on the generators, where ι is the involution which reverses the letters in each word (this is an isomorphism from a free algebra to its opposite). The statement readily follows if we take $\bar{\varepsilon}_i$ to be defined by

$$\bar{\varepsilon}_i(c) := (\varepsilon_i(c))^t,$$

the latter denoting the transpose of a matrix in $M_n(R)$ (this is also an involution inducing an isomorphism from the ring of matrices to its opposite ring). \square

7. Directed systems and consistent sequences of DGAs

Both directed systems and consistent sequences of differential graded algebras appear naturally in applications. In this section we discuss briefly how our constructions can be carried over to these cases.

7.1. The infinitely generated case: a directed system of DGAs

The differential graded algebra considered up to this point have all been finitely generated. For many of the applications that we have in mind this is also sufficient. Namely, the Chekanov-Eliashberg DGA of a Legendrian submanifold Λ is generated by the Reeb chords of Λ , and a generic Legendrian submanifold has finitely many Reeb chords in most contact manifolds for which the Chekanov-Eliashberg DGA is rigorously defined. For example, this is the case for closed Legendrian submanifolds of the standard contact \mathbb{R}^{2n+1} .

Nonetheless, for a general contact manifold there may be infinitely many Reeb chords on a generic closed Legendrian submanifold. In this case then the Chekanov-Eliashberg DGA is infinitely generated, and hence M is a free A - A -bimodule with an infinite preferred basis. However, note that to every Reeb chord we can associate an *action* $\ell \in \mathbb{R}_{>0}$, and generically all Reeb chords below a certain action still comprise a finite subset. We write $M^\ell \subset M$ for the free and finitely generated A - A -bimodule spanned by the Reeb chords of action less than $\ell > 0$. We write $\mathcal{A}^\ell := \mathcal{T}_A(M^\ell)$, and the action-decreasing property of the differential in the Chekanov-Eliashberg DGA implies that each \mathcal{A}^ℓ is a sub-DGA of \mathcal{A} , and therefore there is an induced directed system

$$i_{\ell_1, \ell_2}: (\mathcal{A}^{\ell_1}, \partial) \hookrightarrow (\mathcal{A}^{\ell_2}, \partial), \quad \ell_1 \leq \ell_2,$$

of finitely generated differential graded algebras. The direct limit of this directed system is the infinitely generated differential graded algebra (\mathcal{A}, ∂) and therefore we can reduce the study of an infinitely generated graded algebra endowed with an ‘‘action filtration’’ as above to the study of directed systems of finitely generated DGAs.

In this setting the A_∞ -categories obtained by applying the constructions in Subsections 4.1 and 4.2 to the direct system $(\mathcal{A}^\ell, \partial)$ form an inverse system; namely we have morphisms

$$\begin{aligned} i_{\ell_1, \ell_2}^\vee: (M^{\ell_2})^\vee &\rightarrow (M^{\ell_1})^\vee, & \ell_1 < \ell_2, \\ i_{\ell_1, \ell_2}^*: (M^{\ell_2})^* &\rightarrow (M^{\ell_1})^*, & \ell_1 < \ell_2. \end{aligned}$$

Using the given choice of basis of M , the adjoint morphisms i_{ℓ_1, ℓ_2}^\vee and i_{ℓ_1, ℓ_2}^* both correspond to canonical projections onto the submodules spanned by the generators having actions at most ℓ_1 . The linearised coboundary maps $\mu^{\varepsilon_0, \varepsilon_1}$ (defined using either of the constructions) makes the above inverse systems into inverse systems of complexes; i.e. the above projection maps are chain maps. The Mittag-Leffler property be seen to hold for

the corresponding inverse system of boundaries, and hence the inverse limits of homologies is equal to the homology of the inverse limit complex.

The respective A_∞ -structures constructed for the above inverse system of complexes can then be seen to satisfy $i_{\ell_1, \ell_2}^\vee \circ \mu_n = \mu_n \circ ((i_{\ell_1, \ell_2}^\vee)^{\otimes n})$ and $i_{\ell_1, \ell_2}^* \circ \mu_n = \mu_n \circ ((i_{\ell_1, \ell_2}^*)^{\boxtimes n})$. This gives rise to an A_∞ -structure on the inverse limits of $(M^\ell)^\vee$ and $(M^\ell)^*$.

7.2. Consistent sequences of DGAs

The construction of \mathfrak{A}_n and \mathcal{A}_{A_n} in Section 3.4 out of \mathcal{A} leads to families of differential graded algebras with an increasing number of generators. Such sequences were used in [2] to upgrade the A_∞ algebra structure from [8] to an A_∞ category whose objects are the augmentations of \mathcal{A} . This idea was later generalised in [29], where the notion of a *consistent* family of differentiable graded algebras was introduced. Here we briefly describe this notion and show how it also gives rise to A_∞ -categories with noncommutative coefficients. The geometrical construction underlying this algebraic definition will be sketched in Appendix A.2.

Let (\mathcal{A}, ∂) be a semifree differential graded algebra over the noncommutative algebra A . Its underlying algebra is thus the tensor algebra $\mathcal{T}_A(M)$ over A of a free A - A -bimodule M with basis \mathcal{B} . An (m -components) *link grading* (as introduced in [27]) on \mathcal{A} is a pair of maps $b, e: \mathcal{B} \rightarrow \{1, \dots, m\}$ such that:

- If $c \in \mathcal{B}$ is such that $b(c) \neq e(c)$ then $\partial(c)$ has no constant term, and
- For any $c \in \mathcal{B}$ and any word $a_0 c_1 a_1 \cdots a_{n-1} c_n a_n$ appearing in an expression of $\partial(c)$, we have $b(c_{i-1}) = e(c_i)$.

A generator $c \in \mathcal{B}$ is called *pure* if $b(c) = e(c)$ and *mixed* otherwise. On \mathfrak{A}_n and \mathcal{A}_{A_n} there is a link grading defined by $b(c^{ij}) = i$ and $e(c^{ij}) = j$. Moreover, words $a_0 c_1 a_1 \cdots a_{n-1} c_n a_n$ such that $b(c_{i-1}) = e(c_i)$ (i.e. of the type appearing in the differential of a basis element) are called *composable* in [3], [2] and [29]. This terminology comes from the Chekanov-Eliashberg algebra of an m -components Legendrian link: the components are labeled by $\{1, \dots, m\}$, and the maps b and e give the label of the component of the starting point and endpoint of a Reeb chord of the link. Composable words are those which can appear as negative asymptotics of a holomorphic disc with boundary on the cylinder over the link.

Let (\mathcal{A}, ∂) be a differential graded algebra equipped with a link grading (b, e) and let I be a subset of $\{1, \dots, m\}$. We denote by \mathcal{A}_I the subalgebra of \mathcal{A} generated by basis elements c for which $b(c), e(c) \in I$. There is a projection $\pi: \mathcal{A} \rightarrow \mathcal{A}_I$ such that, for every basis element, $\pi(c) = c$ if $(b(c), e(c)) \in I \times I$, and $\pi(c) = 0$ otherwise. It follows from the definition of a link grading that ∂ descends to a differential $\partial_I = \pi \circ \partial$ on \mathcal{A}_I . For m -components Legendrian links this corresponds to taking chords of the sub-link whose components are labeled by I and defining a differential which counts only holomorphic discs which are asymptotic to chords in this sublink. Note that the differential graded algebra \mathcal{A}_I is equipped with a link grading once we identify I with $\{1, \dots, l\}$ by an order preserving identification. When $I = \{i\}$ we denote \mathcal{A}_I simply by \mathcal{A}_i .

We give now the definition of a consistent family of differential graded algebras following [29].

Definition 7.1. A sequence $(\mathcal{A}^{(i)}, \partial^{(i)})$ of semi-free differential graded algebras with generating sets $\mathcal{B}^{(i)}$ and link gradings $(b^{(i)}, e^{(i)})$ taking values in $\{1, \dots, i\}$ is *consistent* if the following properties are satisfied.

- (1) For every increasing map $f: \{1, \dots, i\} \rightarrow \{1, \dots, j\}$ there is an induced map $h_f: \mathcal{B}^{(i)} \rightarrow \mathcal{B}^{(j)}$ such that, for any generator $c \in \mathcal{B}^{(i)}$, we have

$$(b^{(j)}(h_f(c)), e^{(j)}(h_f(c))) = (f(b^{(i)}(c)), f(e^{(i)}(c))).$$

- (2) For any two composable increasing maps f and g between finite sets, we have $h_{f \circ g} = h_f \circ h_g$.
- (3) For any f as above, the algebra morphism $h_f: \mathcal{A}^{(i)} \rightarrow \mathcal{A}^{(j)}$ coinciding with h_f on generators satisfies the property that $\pi \circ h_f: \mathcal{A}^{(i)} \rightarrow \mathcal{A}_{f(\{1, \dots, i\})}^{(j)}$ is an isomorphism of differential graded algebras.

Note that increasing maps from $\{1, \dots, i\}$ to $\{1, \dots, j\}$ are in one-to-one correspondence with subsets of $\{1, \dots, j\}$ of cardinality i . Figure 2 shows the geometric meaning of the maps h_f when $\mathcal{A}^{(i)}$ is the Chekanov-Eliashberg of the i -copy link of a Legendrian submanifold. See Appendix A.2.2.

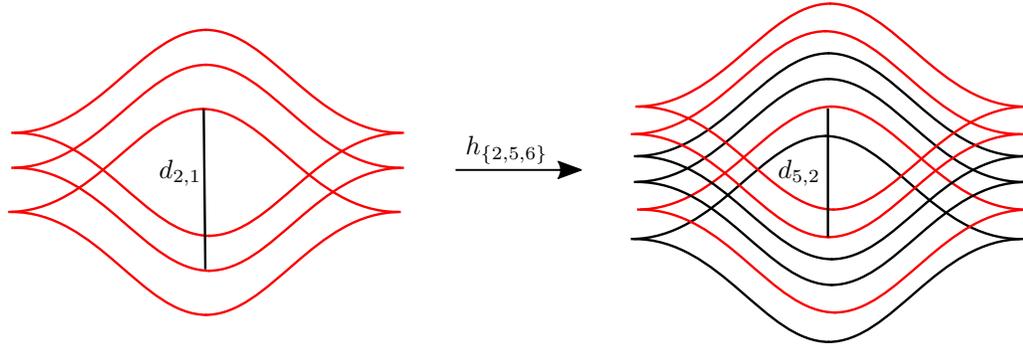


FIGURE 2. The 3-copies link as a sublink of the 6-copy link.

The upshot of this definition is that, since $\mathcal{A}^{(1)}$ is isomorphic to $\mathcal{A}_k^{(i+1)}$ for any k in $\{1, \dots, i+1\}$, any $(i+1)$ -tuple of augmentations $(\varepsilon_0, \dots, \varepsilon_i)$ of $\mathcal{A}^{(1)}$ gives rise to an augmentation ε of $\mathcal{A}^{(i+1)}$ which vanishes on the mixed generators and satisfies $\varepsilon(c) = \varepsilon_k(c)$ for any $c \in \mathcal{A}_k^{(i+1)} \cong \mathcal{A}^{(1)}$.

We denote by $M^{(i+1)}$ the free bimodule generated by $\mathcal{B}^{(i+1)}$. Also for any subset I of $\{1, \dots, j\}$ of cardinality $i+1$, we denote by M^I the corresponding submodule of $M^{(j)}$

(which is identified with $M^{(i+1)}$). We decompose each differential $\partial^{(i+1)}$ restricted to $M^{(i+1)}$ into a sum

$$\partial^{(i+1)}|_{M^{(i+1)}} = \partial_0^{(i+1)} + \cdots + \partial_k^{(i+1)},$$

where $\partial_i^{(i+1)}$ takes values in $(M^{(i+1)})^{\boxtimes i}$. Now, given an $(i+1)$ -tuple of augmentations of $\mathcal{A}^{(1)}$, inducing an augmentation ε of $\mathcal{A}^{(i+1)}$, we define the operation $\mu_i^{\varepsilon_0, \dots, \varepsilon_i}$ as follows. We consider the map

$$\begin{array}{ccc} M^{(2)} & \xrightarrow{h_{\{1, i+1\}}} M^{(i+1)} & \xrightarrow{(\partial^{(i+1)})_\varepsilon} (M^{(i+1)})^{\boxtimes i} \\ & & \downarrow \pi \\ & & M_{\{i, i+1\}}^{(i+1)} \boxtimes M_{\{i-1, i\}}^{(i+1)} \boxtimes \cdots \boxtimes M_{\{1, 2\}}^{(i+1)} \\ & & \downarrow \simeq \\ & & (M^{(2)})^{\boxtimes i}. \end{array} \quad (15)$$

The map $\pi: (M^{(i+1)})^{\boxtimes i} \rightarrow M_{\{i, i+1\}}^{(i+1)} \boxtimes \cdots \boxtimes M_{\{1, 2\}}^{(i+1)}$ is the restriction of the canonical projection $\pi: \mathcal{A}^{(i+1)} \rightarrow \mathcal{A}_{\{i, i+1\}}^{(i+1)} \star \cdots \star \mathcal{A}_{\{1, 2\}}^{(i+1)}$.

This allows us to define A_∞ -categories whose objects are augmentations of $\mathcal{A}^{(1)}$, the morphism space between any pair of augmentation is a copy of $M^{(2)}$, and the compositions are defined by taking adjoints of the maps $M^{(2)} \rightarrow (M^{(2)})^{\boxtimes i}$ defined in (15), using the construction from either Subsection 4.1 or 4.2.

Note that the procedure described in Section 3.4 which associates to \mathcal{A} and n the differential graded algebra \mathfrak{A}_n produces a consistent sequence of differential graded algebras whose link grading is $(b(c^{ij}), e(c^{ij})) = (i, j)$. This sequence has the property that $M^{(2)} \cong M^{(1)} = M$, and therefore the augmentation category defined from it contains the same information as the differential graded algebra \mathcal{A} . However, there exist consistent sequences containing strictly more information than simply that contained in $\mathcal{A} = \mathcal{A}^{(1)}$. For instance, even in the case when \mathcal{A} is finitely generated, an infinite consistent sequence may still give rise an A_∞ -category with nontrivial operations of arbitrarily high order. In Appendix A.2 we sketch the geometric construction of [29], which illustrates such a phenomenon.

Appendix A. The geometric setting

In this appendix, we discuss the geometric motivation of our constructions.

A.1. The Legendrian contact homology with twisted coefficients

Here we give a very brief sketch of the construction of the differential graded algebras that arise in Legendrian contact homology. We refer to [6] and [15] for more details. Let $\Lambda \subset (Y, \alpha)$ be a Legendrian submanifold in a manifold Y with a contact one-form α . The contact one-form induces the Reeb vector field on Y . A *Reeb chord* of Λ is an integral

curves of the Reeb vector field with both endpoints on Λ . Generically, the Reeb chords form a discrete set.

We denote $M_R(\Lambda)$ the free graded R -module generated by the Reeb chords of Λ . The grading is induced by the Conley-Zehnder index of the chords and, in general, takes values in a cyclic group. For now on we assume that there is only a finite number of Reeb chords.

In the most basic setting, Legendrian contact homology associates a differential graded algebra structure on the tensor algebra $\mathcal{T}_R(M_R(\Lambda))$ over R ; this is the Chekanov-Eliashberg DGA. The differential of a Reeb chord d_0 counts the rigid pseudoholomorphic punctured discs in the symplectisation $\mathbb{R} \times Y$, having boundary on the Lagrangian cylinder $\mathbb{R} \times \Lambda$ over Λ and one positive strip-like end asymptotic to d_0 . For example, a pseudoholomorphic disc as shown in Figure 1 gives a contribution to the coefficient in front of the word $d_1 d_2 d_3 d_4 d_5 \in (M_R(\Lambda))^{\otimes 5}$ in the expression of $\partial(d_0)$.

In [19] Eriksson-Östman extends the definition of the Chekanov-Eliashberg DGA to a version with coefficients in the group ring $A := R[\pi_1(\Lambda)]$. In this case the underlying graded algebra is the tensor algebra $\mathcal{T}_A(M_A(\Lambda))$, where $M_A(\Lambda)$ is the free graded A - A -bimodule generated by the Reeb chords of Λ . Fix a choice of capping paths for each end-point of a Reeb chord, i.e. a path in Λ connecting the end-point with a given base point. Roughly speaking, the differential then takes the following form. Assume that $a_0, a_1, \dots, a_5 \in \pi_1(\Lambda)$ in Figure 1 denote the homotopy classes of closed curves corresponding to the canonically oriented boundary arcs in $\mathbb{R} \times \Lambda$ of a pseudoholomorphic disc contributing to the differential, where each boundary arc has been closed up by using the corresponding capping path. Then the depicted disc contributes to the R -coefficient in front of $a_0 d_1 a_1 d_2 a_2 d_3 a_3 d_4 a_4 d_5 a_5 \in (M_A(\Lambda))^{\boxtimes 5}$ in the expression $\partial(d_0)$.

Remark A.1. Note that the above construction requires that we introduce auxiliary capping paths in Λ connecting each end-point of a Reeb chord with a given base point. It would be more natural to replace the fundamental group by the fundamental groupoid, and consider differential graded algebras over a groupoid algebra. This generalisation does not require any new idea, but for simplicity of notation we will only consider group algebras.

A.2. \mathcal{A}_∞ -categories of a Legendrian link.

The construction of Section 7.2 allows us to define several versions of \mathcal{A}_∞ -categories coming from various consistent sequences of DGAs.

A.2.1. The “negative” augmentation and representation categories.

Let Λ be a Legendrian submanifold such that its Chekanov-Eliashberg DGA (over the appropriate coefficient algebra) admits augmentations. Then, by applying the constructions of Subsections 4.1 and 4.2, we associate to Λ three \mathcal{A}_∞ -categories which generalise the augmentation category of [2] to the noncommutative setting. In the terminology of [29] these are the “negative” augmentation categories.

The category $\mathcal{A}ug_-(\Lambda, R[\pi_1(\Lambda)])$. Let $\mathcal{A}(\Lambda)$ be the Chekanov-Eliashberg algebra of Λ as defined in Appendix A.2 with coefficients in $R[\pi_1(\Lambda)]$. We define $\mathcal{A}ug_-(\Lambda, R[\pi_1(\Lambda)])$ as the \mathcal{A}_∞ -category such that:

- (1) $\mathcal{O}b(\mathcal{A}ug_-(\Lambda, R[\pi_1(\Lambda)]))$ is the set of augmentations $\varepsilon: \mathcal{A} \rightarrow R[\pi_1(\Lambda)]$.
- (2) For every pair of augmentations $\varepsilon_1, \varepsilon_2 \in \mathcal{O}b(\mathcal{A}ug_-(\Lambda, R[\pi_1(\Lambda)]))$, the morphism space $\text{Hom}(\varepsilon_1, \varepsilon_2)$ is the free $R[\pi_1(\Lambda)]$ - $R[\pi_1(\Lambda)]$ -bimodule generated by the Reeb chords of Λ .
- (3) The operations μ_n , $n \geq 1$, are the $R[\pi_1(\Lambda)]$ - $R[\pi_1(\Lambda)]$ -bimodule maps

$$\mu_n: \text{Hom}(\varepsilon_{n-1}, \varepsilon_n) \boxtimes \text{Hom}(\varepsilon_{n-2}, \varepsilon_{n-1}) \boxtimes \cdots \boxtimes \text{Hom}(\varepsilon_0, \varepsilon_1) \rightarrow \text{Hom}(\varepsilon_0, \varepsilon_n)$$

defined in Subsection 4.2.

The fact that $\mathcal{A}ug_-(\Lambda, R[\pi_1(\Lambda)])$ is an \mathcal{A}_∞ -category comes from Theorem 4.11. Let $d_2 \in \text{Hom}(\varepsilon_2, \varepsilon_3)$ and $d_4 \in \text{Hom}(\varepsilon_1, \varepsilon_2)$. Assuming that $\varepsilon_2(d_3) = \sum a_k h_k$, the disc in Figure 3 gives a contribution of $a_k g_1^{-1} \varepsilon_3(d_1) * g_0^{-1} d_0 g_5^{-1} \varepsilon_1(d_5) * g_4^{-1}$ to $\mu_2(d_2 h_k d_4)$ for all k .

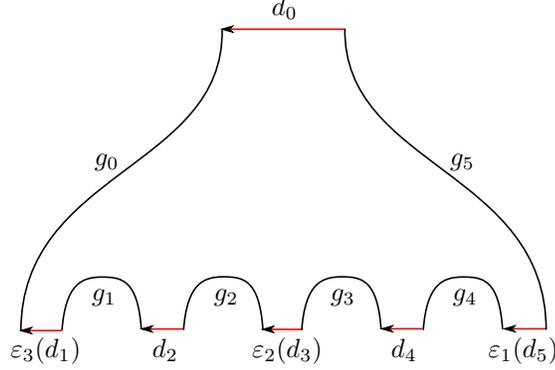


FIGURE 3. A pseudoholomorphic disc contributing to the map $\mu_2: \text{Hom}(\varepsilon_2, \varepsilon_3) \boxtimes \text{Hom}(\varepsilon_1, \varepsilon_2) \rightarrow \text{Hom}(\varepsilon_1, \varepsilon_3)$.

The category $\mathcal{R}ep(\Lambda, m)$. Let $\mathcal{A}(\Lambda)$ be the Chekanov-Eliashberg algebra of Λ with coefficients in R . For $m \in \mathbb{N}$, we denote by $M_m(R)$ the algebra of $m \times m$ matrices with entries in R . We define $\mathcal{R}ep(\Lambda, m)$ as the \mathcal{A}_∞ -category such that:

- (1) $\mathcal{O}b(\mathcal{R}ep(\Lambda, m))$ is the set of augmentations $\rho: \mathcal{A} \rightarrow M_m(R)$ (called m -dimensional representations of Λ).
- (2) For every pair of augmentations $\rho_1, \rho_2 \in \mathcal{O}b(\mathcal{R}ep(\Lambda, m))$, the morphism space $\text{Hom}(\rho_1, \rho_2)$ is the free $M_m(R)$ - $M_m(R)$ -bimodule generated by Reeb chords of Λ .
- (3) The operations μ_n , $n \geq 1$, are the $M_m(R)$ - $M_m(R)$ -bimodule maps

$$\mu_n: \text{Hom}(\rho_{n-1}, \rho_n) \boxtimes \text{Hom}(\rho_{n-2}, \rho_{n-1}) \boxtimes \cdots \boxtimes \text{Hom}(\rho_0, \rho_1) \rightarrow \text{Hom}(\rho_0, \rho_n)$$

defined in Subsection 4.2.

From Theorem 4.11 it follows that $\mathcal{R}\text{ep}(\Lambda, m)$ is an \mathcal{A}_∞ -category. Assuming that $\rho_2(d_3)$ is the matrix with coefficients (a_{ij}) and $E_{i,j}$ denotes the elementary matrix with entries $(\delta_{i,j})$, the pseudoholomorphic disc in Figure 4 gives a contribution of $a_{ij}\rho_3(d_1)^*d_0\rho_1(d_5)^*$ to $\mu_2(d_2E_{i,j}d_4)$ for all i, j .

The category $\mathcal{R}\text{ep}_-(\Lambda, S)$. Given a noncommutative R -algebra S , we define the category $\mathcal{R}\text{ep}(\Lambda, S)$ such that:

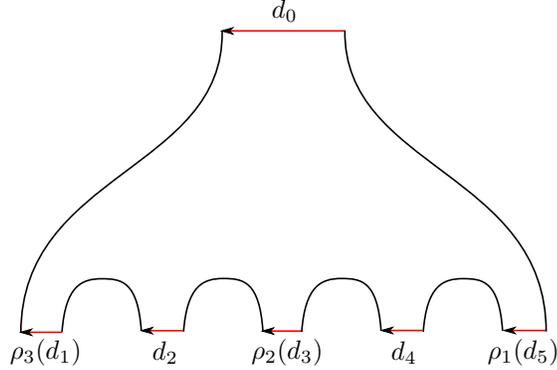


FIGURE 4. A pseudoholomorphic disc contributing to the map $\mu_2: \text{Hom}(\rho_2, \rho_3) \boxtimes \text{Hom}(\rho_1, \rho_2) \rightarrow \text{Hom}(\rho_1, \rho_3)$.

- (1) $\mathcal{O}b(\mathcal{R}\text{ep}(\Lambda, S))$ consists of augmentations $\rho: \mathcal{A}(\Lambda) \rightarrow S$.
- (2) For every pair of augmentations $\rho_1, \rho_2 \in \mathcal{O}b(\mathcal{R}\text{ep}(\Lambda, m))$, the space of morphisms $\text{Hom}(\rho_1, \rho_2)$ is the free S -module generated by the Reeb chords of Λ
- (3) The operations $\mu_n, n \geq 1$, are the R -linear maps

$$\mu_n: \text{Hom}(\rho_{n-1}, \rho_n) \otimes \text{Hom}(\rho_{n-2}, \rho_{n-1}) \otimes \cdots \otimes \text{Hom}(\rho_0, \rho_1) \rightarrow \text{Hom}(\rho_0, \rho_n)$$

defined in Subsection 4.1.

We can also make hybrid constructions where the Chekanov-Eliashberg algebra is defined over the group ring of $\pi_1(\Lambda)$ and the augmentations take values in a matrix algebra $M_m(R)$ or in a more general noncommutative algebra S .

A.2.2. The “positive” augmentation and representation categories.

The consistent sequence leading to $\mathcal{A}\text{ug}_-(\Lambda)$ is determined uniquely by $\mathcal{A}(\Lambda)$ as described in [2] (see the discussion at the end of Section 7.2) and does not require the language of consistent sequences of DGAs. However, additional geometric input is needed in order to define $\mathcal{A}\text{ug}_+$. For a Legendrian submanifold Λ in the jet space $J^1(Q)$, we denote by Λ_n the Legendrian link obtained by taking a generic small perturbation of n -copies of Λ , which we denote by $\Lambda_1, \dots, \Lambda_n$, shifted in the direction of the Reeb vector field by less than the length of the smallest Reeb chords of Λ divided by n . Note that for any chord c

of Λ and pair (i, j) with $i, j \in \{1, \dots, n\}$ there is a corresponding chord $d_{i,j}$ of Λ_n starting on the i -th copy of Λ and ending on the j -th copy.

The description of the consistent sequence requires some care in the choice of the perturbation of the n -copy link. Let $\{f_n\}_{n \in \mathbb{N}}$ be a family of Morse functions on S^1 each of which has only two critical points m_n and M_n and satisfies the following conditions:

- (1) for any n , there is an inclusion of the oriented intervals $(M_{n+1}, m_{n+1}) \subset (M_n, m_n)$ (i.e. the critical points are *nested*).
- (2) For any $i < j$, $f_i - f_j$ is a Morse function with only two critical points: one maximum contained in (M_i, M_j) and one minimum in (m_j, m_i) .

Remark A.2. The existence of sequences of Morse functions whose critical points share similar combinatorics is one technical point that need be resolved to extend the definition of $\mathcal{A}ug_+$ to Legendrian submanifolds of high dimensions. Some notion of coherent system of perturbations in the spirit of (but not exactly like) [33] would allow one to extend this setup and get a well defined category with the correct invariance properties.

Using this perturbation, any chord of the i -copy link becomes a chords of the j -copy link when $\{1, \dots, i\}$ is a subset of $\{1, \dots, j\}$. The Chekanov-Eliashberg DGAs associated to this sequence of Legendrian links form a consistent sequence of differential graded algebras where $\mathcal{A}^{(i)}$ is the Chekanov-Eliashberg DGA of the link Λ_i . The link grading is defined by taking the beginning and end of Reeb chords; see Figure 2.

This consistent sequence of differential graded algebra allows us to define “plus” variants of the previous “minus” augmentation categories that we now proceed to outline.

The categories $\mathcal{A}ug_+(\Lambda, R[\pi_1(\Lambda)])$ and $\mathcal{R}ep_+(\Lambda, n)$. The objects are the same as in $\mathcal{A}ug_-(\Lambda, R[\pi_1(\Lambda)])$ and $\mathcal{R}ep(\Lambda, n)$ but the morphisms space are generated by Reeb chords of Λ and critical points of f_1 . They can be identified to mixed chords in Λ_2 and therefore the morphism spaces correspond to $M^{(2)}$ in the language of Section 7.2. The operations are defined by applying the dualisation process of subsection 4.2 to the maps given by equation (15).

Figures 3 and 4 show again how to compute such operation: note that in this situation the (non-augmented) chords d_2 and d_4 are mixed chords of the 3-copy link and can come from critical points of the functions $f_i - f_j$, the remaining (augmented) chords d_1, d_3 and d_5 are pure chords.

The category $\mathcal{R}ep_+(\Lambda, S)$. This is a similar generalisation of $\mathcal{R}ep_-(\Lambda, S)$. Again the contribution of a pseudoholomorphic disc is as shown in Figure 1, with the understanding that possibly some non-augmented chords are mixed (and possibly come from Morse critical points).

Remark A.3. It follows from the argument in [29] that the “plus” categories admits a strict unit (for some particular choices of almost-complex structures), where the unit is given by m . Though this unit is not necessarily closed in the cases $\mathcal{A}ug_+(\Lambda, R[\pi_1(\Lambda)])$ and $\mathcal{R}ep_+(\Lambda, n)$ (Case II i.e. Section 4.2), it can be seen to be closed in the case of $\mathcal{R}ep_+(\Lambda, S)$ (Case I i.e. Section 4.1); see [16, Theorem 5.5]. For instance, in the case of $\mathcal{R}ep(\Lambda, n)$, it follows from the description of holomorphic disks having negative ends asymptotic to

the minimum chord m that the boundary of m is equal to $\sum_c \rho(c)^T \cdot m - m \cdot \rho(c)^T$. Due to the noncommutativity, the latter expression is possibly nonvanishing whenever $\rho(c)$ is. Note that the construction in Section 4.1 (Case I) still makes the dual of m a cycle in this case.

Remark A.4. It is possible to define $\mathcal{A}ug_-$ by noting that if (b, e) is a link grading then (e, b) is also a link grading. Applying this change to the link grading of λ_n , the consistent sequence leading to $\mathcal{A}ug_+$ gives the consistent sequence leading to $\mathcal{A}ug_-$ because Reeb chords corresponding to critical points if the Morse functions go in the wrong direction.

A.3. A note about invariance

The invariance properties satisfied by the constructions carried out in this paper will not be discussed in detail. Chekanov-Eliashberg DGA’s are invariant up to so-called “stable-tame isomorphism”. From this it is not difficult to see that the set of isomorphism classes of linearised homologies is invariant, as it was originally shown in [6]. The fact that the coefficients are noncommuting plays no important role in that proof, and therefore the same result holds in the current setting as well.

Similarly the dual complexes constructed in Subsections 4.1 and 4.2 satisfy the following invariance property. Consider the so-called “bilinearised co-complexes” $(M^\vee, \mu_1^{(\varepsilon_1, \varepsilon_2)})$ from Subsection 4.1 or $(M, \mu_1^{(\varepsilon_1, \varepsilon_2)})$ from Subsection 4.2. The isomorphism classes of their homologies for all possible pairs of augmentations $(\varepsilon_1, \varepsilon_2)$ of \mathcal{A} into A is then invariant under stable-tame isomorphism of the differential graded algebra \mathcal{A} . In fact, since homotopic augmentations (in the sense of DGA morphisms) induce the same bilinearised (co)complex, this set of isomorphism classes is even invariant under DGA homotopies (see [22, Chapter 26] for the definition). We refer to [1, Theorem 2.8] for the proof of a similar statement.

For the invariance properties of the augmentation A_∞ -categories we refer to [2], which handles the case when A is commutative. It is shown there that the A_∞ -categories associated to two stable-tame isomorphic DGAs are “pseudo equivalent” (see the mentioned article for this notion). The general case follows similarly. Finally, the invariance of $\mathcal{A}ug_+(\Lambda)$ up to quasi-equivalence follows again from the stable tame isomorphism class of $\mathcal{A}(\Lambda)$ as shown in [29] and can be easily generalised to noncommutative coefficients.

References

- [1] F. Bourgeois. A survey of contact homology. In *New perspectives and challenges in symplectic field theory*, volume 49 of *CRM Proc. Lecture Notes*, pages 45–71. Amer. Math. Soc., Providence, RI, 2009.
- [2] F. Bourgeois and B. Chantraine. Bilinearized Legendrian contact homology and the augmentation category. *J. Symplectic Geom.*, 12(3):553–583, 2014.
- [3] F. Bourgeois, T. Ekholm, and Y. Eliashberg. Effect of Legendrian surgery. *Geom. Topol.*, 16(1):301–389, 2012.
- [4] H. Cartan. Sur les groupes d’Eilenberg-MacLane h (II, n): I. Methode des constructions. *Proceedings of the National Academy of Sciences of the United States of America*, 40(6):467, 1954.

- [5] B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini, and R. Golovko. Floer theory for Lagrangian cobordisms. Preprint (2015), available at <http://arxiv.org/abs/1511.09471>.
- [6] Yu. V. Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002.
- [7] C.-H. Cho, H. Hong, and L. Siu-Cheong. Noncommutative homological mirror functor. Preprint (2015), available at <http://arxiv.org/abs/1512.07128>.
- [8] G. Civan, P. Koprowski, J. Etnyre, J. M. Sabloff, and A. Walker. Product structures for Legendrian contact homology. *Math. Proc. Cambridge Philos. Soc.*, 150(2):291–311, 2011.
- [9] O. Cornea and F. Lalonde. Cluster homology. Preprint (2005), available at <http://arxiv.org/abs/0508345>.
- [10] M. Damian. Floer homology on the universal cover, Audin’s conjecture and other constraints on Lagrangian submanifolds. *Comment. Math. Helv.*, 87(2):433–462, 2012.
- [11] G. Dimitroglou Rizell. Nontriviality results for the characteristic algebra of a DGA. Preprint (2015), available at <http://arxiv.org/abs/1512.03570>.
- [12] G. Dimitroglou Rizell. Exact Lagrangian caps and non-uniruled Lagrangian submanifolds. *Arkiv för Matematik*, pages 1–28, 2014.
- [13] G. Dimitroglou Rizell and R. Golovko. Estimating the number of Reeb chords using a linear representation of the characteristic algebra. *Algebr. Geom. Topol.*, 15(5):2887–2920, 2015.
- [14] D. S. Dummit and R. M. Foote. Abstract algebra, 3rd edition, 2004.
- [15] T. Ekholm, J. Etnyre, and M. Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. *Trans. Amer. Math. Soc.*, 359(7):3301–3335 (electronic), 2007.
- [16] T. Ekholm, J. B. Etnyre, and J. M. Sabloff. A duality exact sequence for Legendrian contact homology. *Duke Math. J.*, 150(1):1–75, 2009.
- [17] T. Ekholm and L. Ng. Legendrian contact homology in the boundary of a subcritical weinstein 4-manifold. *Journal of Differential Geometry*, 101(1):67–157, 9 2015.
- [18] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. *Geom. Funct. Anal.*, (Special Volume, Part II):560–673, 2000. GAFA 2000 (Tel Aviv, 1999).
- [19] A. Eriksson Östman. Legendrian contact homology with homotopy coefficients. In preparation, 2016.
- [20] J. B. Etnyre and K. Honda. On connected sums and Legendrian knots. *Adv. Math.*, 179(1):59–74, 2003.
- [21] J. B. Etnyre, L. L. Ng, and J. M. Sabloff. Invariants of Legendrian knots and coherent orientations. *J. Symplectic Geom.*, 1(2):321–367, 2002.
- [22] Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.
- [23] K. Fukaya. Morse homotopy, A^∞ -category, and Floer homologies. In *Proceedings of GARC Workshop on Geometry and Topology ’93 (Seoul, 1993)*, volume 18 of *Lecture Notes Ser.*, pages 1–102. Seoul Nat. Univ., Seoul, 1993.
- [24] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
- [25] M. Kontsevich. Homological algebra of Mirror Symmetry. *Proceedings of the International Congress of Mathematicians, Zurich, Switzerland 1994*, 1:120–139, 1995.
- [26] J. E. Licata and J. M. Sabloff. Legendrian contact homology in Seifert fibered spaces. *Quantum Topology*, 4(3):265–301, 2013.
- [27] K. Mishachev. The N -copy of a topologically trivial Legendrian knot. *J. Symplectic Geom.*, 1(4):659–682, 2003.
- [28] L. Ng and D. Rutherford. Satellites of Legendrian knots and representations of the Chekanov-Eliashberg algebra. *Algebr. Geom. Topol.*, 13(5):3047–3097, 2013.
- [29] L. Ng, D. Rutherford, V. Shende, S. Sivek, and E. Zaslow. Augmentations are Sheaves. Preprint (2015), available at <http://arxiv.org/abs/1502.04939>.
- [30] L. L. Ng. Computable Legendrian invariants. *Topology*, 42(1):55–82, 2003.

- [31] D. Rutherford. Thurston-Bennequin number, Kauffman polynomial, and ruling invariants of a Legendrian link: the Fuchs conjecture and beyond. *Int. Math. Res. Not.*, pages Art. ID 78591, 15, 2006.
- [32] J. M. Sabloff. Invariants of legendrian knots in circle bundles. *Communications in Contemporary Mathematics*, 5(04):569–627, 2003.
- [33] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [34] Clayton Shonkwiler and David Shea Vela-Vick. Legendrian contact homology and nondestabilizability. *J. Symplectic Geom.*, 9(1):33–44, 2011.
- [35] S. Sivek. The contact homology of Legendrian knots with maximal Thurston-Bennequin invariant. *J. Symplectic Geom.*, 11(2):167–178, 2013.
- [36] J. D. Stasheff. Homotopy associativity of H -spaces. I, II. *Trans. Amer. Math. Soc.* 108 (1963), 275-292; *ibid.*, 108:293–312, 1963.
- [37] M. G. Sullivan. K -theoretic invariants for Floer homology. *Geom. Funct. Anal.*, 12(4):810–872, 2002.

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