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Corks

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Abstract. Remarks relating the various notions of corks.

I recall, my advisor Kirby kindly named the object I had constructed in [A2] "Akbulut cork". Since then, this concept went through some evolutions, resulting some confusions between its various different definitions. To clarify them, I decided to collect here mutually consistent definitions.

Definition: A **loose cork** of a smooth manifold M is a pair (W, f), where W is a smooth contractible submanifold of M with involution $f : \partial W \to \partial W$, such that cutting W out of M then regluing with f changes the smooth structure of M. A **cork** (W, f) is a loose cork, which is a Stein manifold.

We call the operation of cutting W from M and regluing by the map f *a loose cork-twisting operation on* M, or simply cork-twisting operation on M. At first glance the "Stein" condition might look artificial, but it really is not, because by [AM2] every loose cork (W, f) contains a cork, hence cork is a smaller fundamental object. Furthermore, given a contractible manifold with involution on its boundary (W, f), it is a difficult to decide whether W is a loose cork (is there a corresponding M?), or does f extend to a self diffeomorphism of W?. To answer the second question in the negative, it suffices to find a pair of knots $\gamma, \gamma' \subset \partial W$ such that γ is not slice but γ' is slice in W, with $f(\gamma) = \gamma'$. As in the example of Figure 9.1 of [A1] (if γ was slice you would get a violation to the "Adjunction Inequality for Stein manifolds", which had been proven in [AM1]).

To find such examples (W, γ, γ'), symplectic topology can be useful. For example, when W is a Stein manifold, attaching W a 2-handle along a knot $\gamma \subset W$ with $TB(\gamma) - 1$ framing extends the Stein structure from W to $W \smile h^2_\gamma$ (In fact, by [LM] and [AO], we can even compactify any Stein manifold into a closed symplectic manifold which could play the role of M in the above definition of cork). Then by the Adjunction inequality proved for the Stein manifolds in [AM1], applied to $W \smile h^2_\gamma$, shows that γ can not be slice.

Sometimes after I wrote [A2], I realized that, even though the cork, I constructed there, looks like the manifold which Mazur had defined in his Annals paper, it is not. Mazur's manifold is not even a "loose cork"!

Matveyev [M] generalized [A2] by proving any exotic copy M' of a simply connected closed smooth 4-manifold M can be obtained by performing a loose cork-twisting operation on M. Hence by [AM2] it can be obtained by a cork-twisting operation on M. Curtis-Freedman-Hsiang-Stong [CFHS] proved a weaker version of [M] without mentioning a cork, namely "M' can be obtained by cutting out a smooth compact contractible submanifold of M, then gluing another smooth compact contractible manifold with the same boundary".

By weakening the involution condition f on a loose cork (W, f) one can construct an infinite order loose cork; [A3] and [G] are such examples. Notice that in minute 1:14:20 of my talk in [A4] I am trying to construct an infinite order cork (with"Stein" condition), which we still don't know if exists.

References

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